Chapter 7

Torsion

This chapter starts with torsion theory in the circular cross section followed by the behaviour of torsion member. The calculation of the stress stress and the angle of twist will be also showed here. Lastly, the topics that include the members having noncircular cross sections will be discussed. The examples and exercises will be included to better understanding.

After successfully completing this chapter you should be able to:

- Understand torsion theory and their applications
- Analysed and calculate torsion in solid and hollow circular bar
- Analysed and calculate the torsion with end-restraints
- Analysed and calculate the torsion with combined bar

7.1 Introduction

Torsion commonly found in mechanical engineering application, for example machinery structures that has twisting member. On the other hand, in civil engineering applications there are only few structures that are subjected to torsion (some torsion effects are negligible). Among those structures are main beam that bearing load from secondary beam, beam that support water channel, and advert post, which is shown in Figure 7.1(a), 7.1 (b) and 7.1 (c). In Figure 7.1(a), secondary beam will distributes load to main beam as point load and moment at connection part of the beams. The moment that produced in secondary beam will be distributed to main beam as torsion moment.

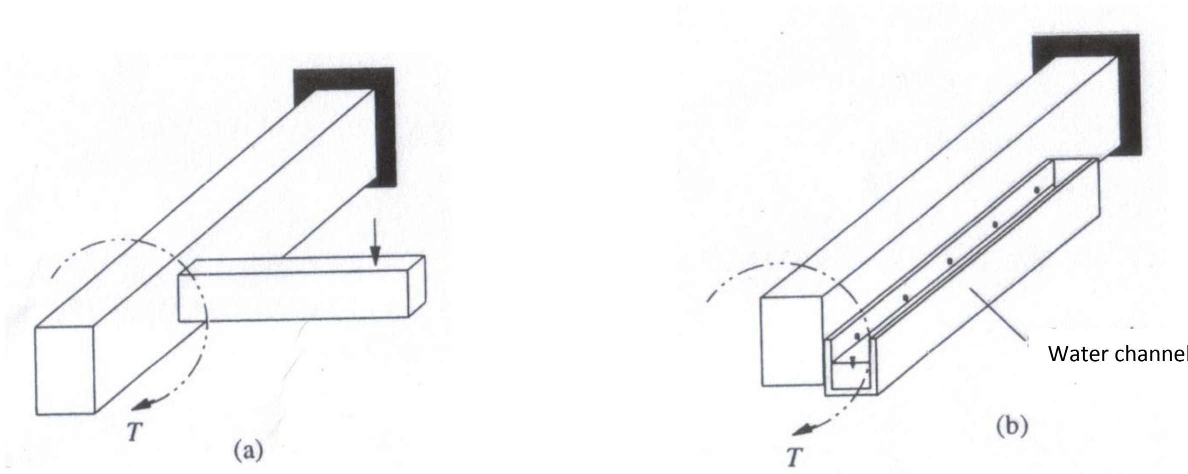


Figure 7.1: Example of structure subjected to torsion

In Figure 7.1 (b), water retaining in a channel will produce moment that are distributed to the beam as torsion. For Figure 7.1 (c), advert post will experience torsion due to wind load that acting on advert planks.

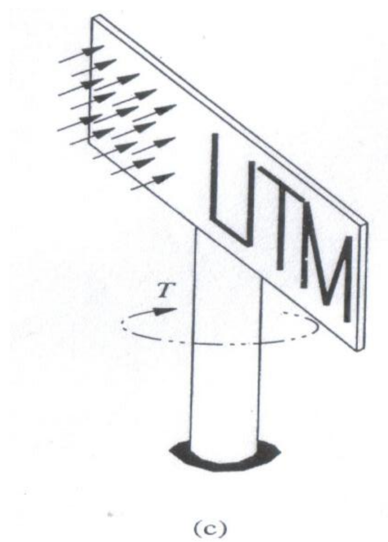


Figure 7.1: Example of structure subjected to torsion (con'd)

7.2 Torsion theory

In previous chapter, we had discussed stress and strain for structures members subjected to axial load. Current chapter discussed structures members subjected to torsion, T . Torsion is twisting that subjected to a structures member that caused the member to twist on member axes.

Considering a solid bar with length *L*, subjected to torsion *T* at its free edge. Therefore, torsion reacted at opposite direction as shown in Figure 7.2.

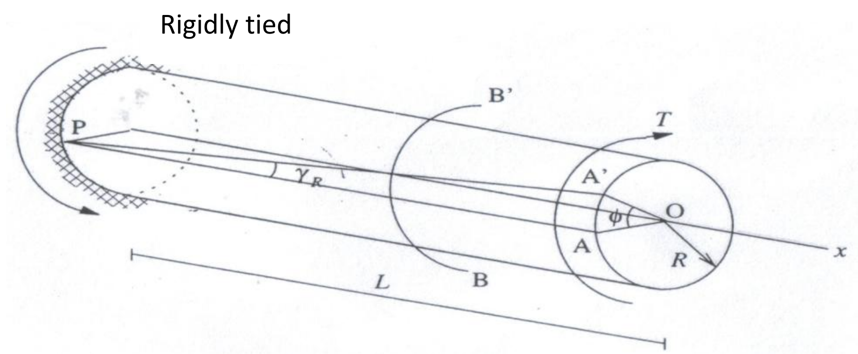


Figure 7.2: A bar subjected to torsion

Using *Right Hand Law*, torsion vector can be determined as shown in Figure 7.3. This torsion may cause compression along the bar given. Few assumptions are taken into account in torsion analysis,

(a) Planar cross-sectional parallel with member axis will remains planar after subjected to torsion.

(b) Shear strain, γ is changing linearly along the bar.

Figure 7.3: Torsion vector

From Hooke’s Law, we are familiar with following relationship,

$\tau = G\gamma$

where τ is shear stress

G is modulus of rigidity

γ is shear strain on the surface

Refer to Figure 7.2, shear strain changing linearly to the axis that passing through bar centroid. From geometry in Figure 7.4 (a), can be written as

$AA' = L\tan\gamma_R = R\tan\phi$

For small angle, therefore $\tan\gamma_R = \gamma_R$ and $\tan\phi = \phi$

$\therefore L\gamma_R = R\varphi \rightarrow \frac{\phi}{L} = \frac{\gamma_R}{R}$

Similarly, shear strain at point with distance r from centroid O as shown in Figure 7.4(b) can be written as,

$L\gamma_r = r\,\phi \rightarrow \frac{\phi}{L} = \frac{\gamma_r}{r}$

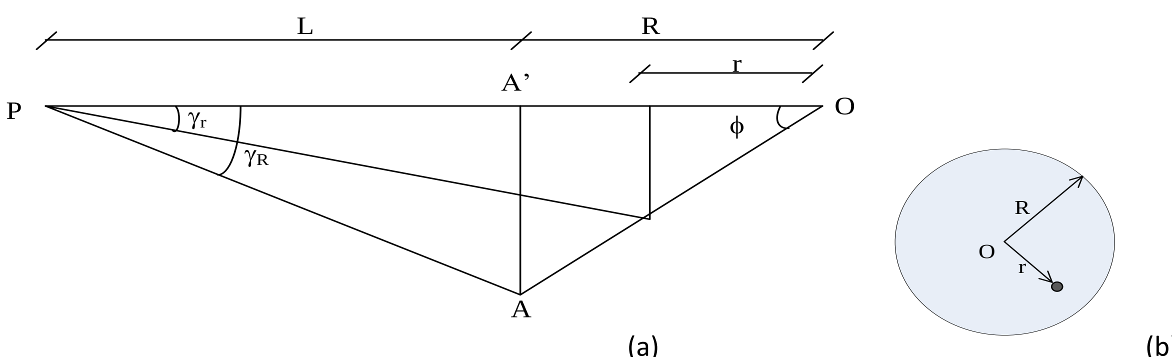


Figure 7.4: Shear strain at radius *R* and *r*

By made equal Eq. (7.1) and Eq. (7.2), therefore

$$\frac{\gamma_r}{r} = \frac{\gamma_R}{R}$$

See a cross-sectional bar as shown in Figure 7.5. Shear stress that produced will be distributed on cross sectional surface and this will produced internal torsion.

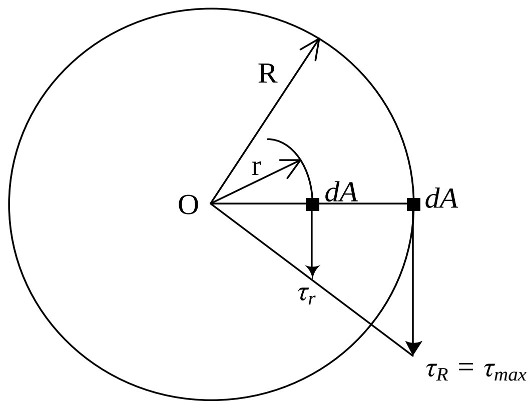


Figure 7.5: Shear stress at radius *R* and *r*

$$\frac{\tau_r}{r} = \frac{\tau_R}{R} \rightarrow \tau_r = \frac{r\tau_R}{R}$$

Shear force that built due to shear stress is

$$dF = \tau_r dA = \frac{r\tau_R}{R}\, dA$$

By taking moment at point O,

$$dM_0 = dF(r) = \frac{r^2\tau_R}{R}\, dA$$

For equilibrium, total moment at point O must be equal to T. Therefore,

$$M_0 = T = \int \frac{r^2\tau_R}{R}\, dA = \frac{\tau_R}{R}\int r^2\, dA$$

Observed that term $\int r^2\, dA$ is polar inertia moment or commonly given as symbol J. Therefore, the equation becomes,

$$T = \frac{\tau_R J}{R} \; or \; \tau_R = \frac{TR}{J} = \tau_{max}$$

Similarly,

$$\tau_r = \frac{Tr}{J}$$

where τ_r is shear stress at any point from centre $\left(\frac{N}{mm}^2\right)$

T is torsion (Nm)

r is radius at certain point from bar centre-point (m)

J is polar inertia moment (m^4)

R is bar radius (m)

For solid circular cross-section as shown in Figure 7.6, J can be obtained as follows,

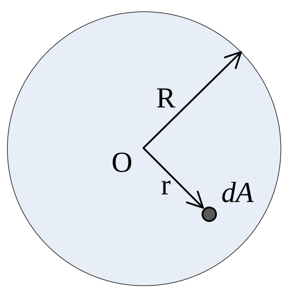


Figure 7.6: Solid circular section

$$J = \int r^2 \, dA$$

$$= \int_0^{2\pi} \int_0^R r^2 \, (r \, d\theta) \, dr$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^R \, d\theta$$

$$= \frac{\pi R^4}{2} = \frac{\pi d^4}{32}$$

Similarly, for circular hollow cross-section as shown in Figure 7.7, J can be obtained from formula,

$$J = \frac{\pi(R_I^4 - R_d^4)}{2} = \frac{\pi(d_I^4 - d_d^4)}{32}$$

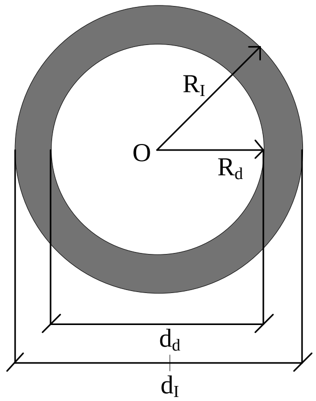
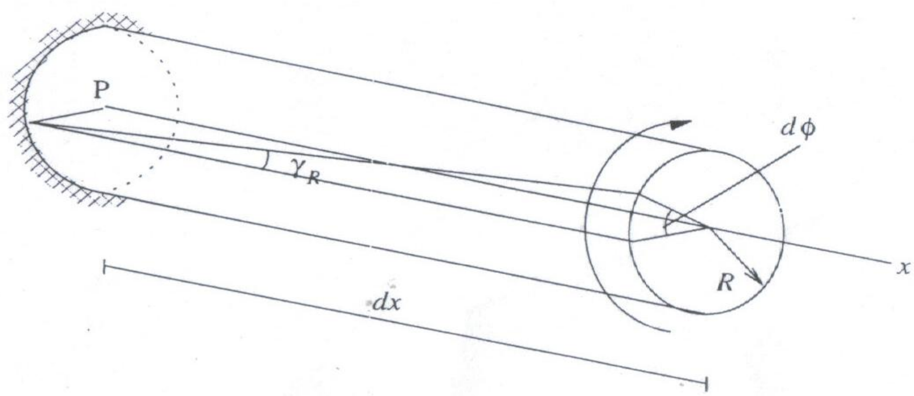


Figure 7.7: Hollow circular section

7.3 Twisting angle

Twisting angle is angle (in radian) produced when a bar is subjected to torsion. Considers a cross-section with distance dx from one bar end as shown in Figure 7.8.



$d\phi$ is twisting angle at distance dx

Figure 7.8: Twisting angle

$\gamma_R dx = R\, d\phi$

$d\phi = \dfrac{\gamma_R dx}{R}$

We knew that $\tau_R = G\gamma_R$

$\therefore d\phi = \dfrac{\tau_R\, dx}{GR}$

Had been derived that $\tau_R = \dfrac{TR}{J}$

$\therefore d\phi = TR\; dx \dfrac{\square}{\square} JGR$

$$= T dx \frac{\square}{\square} JG$$

$$\phi = \int_{x=0}^{x=L} T \frac{dx}{JG}$$

If $T, J, and\ G$ are constant, therefore

$\phi = \dfrac{TL}{JG}$

This equation showed that twisting angle ϕ , is linearly with torsion T , for material in elastic range only

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7.4 Combined Bar

Combined bar consists of two or more materials to form a structure. An example is shown in Figure 7.9. Superposition principle is used to solve this problem.

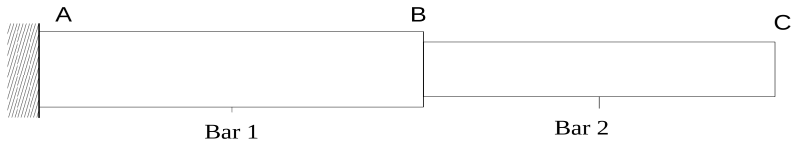


Figure 7.9: Combined bar

The concept to solve combined bar are:

(a) Imposed external torsion is equal to total torsion formed in the bar, i.e.,

$T = T_{bar_1} + T_{bar_2}$ if with two bars

(b) Twisting angle first material is equal to twisting angle of second material at connection part, i.e.,

$\phi_1 = \phi_2$

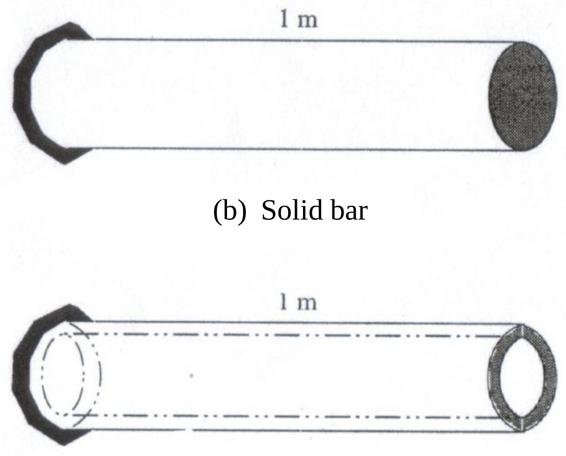
(c) Total twisting angle can be calculated from formula, $\phi = \sum \frac{TL}{JG}$

Example 7.1

Determine torsion, T that imposed to a bar and twisting angle if maximum shear stress is $50 \; x \; 10^3 \; \frac{kN^2}{m}$. Given $G = 80 \; x \; 10^6 \; \frac{kN^2}{m}$.

(a) If solid bar with diameter 0.1 m. See Figure 9.10 (a).

(b) If hollow bar with external and internal and external diameter are 0.1 m and 0.05 m respectively. See Figure 9.10 (b).



(b) Solid bar

(a) Hollow bar

Figure 7.10: Solid and hollow circular bar

Solution

(a) Solid bar

$$J = \frac{\pi R^4}{2} = \frac{\pi\,(0.05)^4}{2} = 9.82\ x\ 10^{-6}\ m^4$$

$$T = \frac{\tau_R J}{R} = \frac{(50000)(9.82\ x\ 10^{-6})}{0.05} = 9.82\ kNm$$

$$\phi = \frac{TL}{JG} = \frac{9.82\ x\ 1.0}{(9.82\ x\ 10^{-6})(80\ x\ 10^6)} = 0.0125\ rad$$

(b) Hollow bar

$$J = \frac{\pi\,(R_{ext}^4 - R_{int}^4)}{2} = \frac{\pi(0.05^4 - 0.025^4)}{2} = 5.64\ x\ 10^{-6}\ m^4$$

$$T = \frac{\tau_R J}{R} = \frac{(50000)(5.64\ x\ 10^{-6})}{0.05} = 5.64\ kNm$$

$$\phi = \frac{TL}{JG} = \frac{5.64\ x\ 1.0}{(5.64\ x\ 10^{-6})(80\ x\ 10^6)} = 0.0125\ rad$$

Example 7.2

Two bars are connected and subjected to torsion as shown in Figure 9.11 (a). Determine the value of maximum shear stress and determine the point of maximum shear. Then, determine the twisting angle at C. Given;

	Radius	Rigidity Modulus
Bar AB	50 mm	$3 \; x \; 10^{10} \; Pa$
Bar BC	25 mm	$8 \; x \; 10^{10} \; Pa$

$$1 \; Pa = 1 \; \frac{N^2}{m}$$

Solution

Determine polar inertia moment,

$$J_{AB} = \frac{\pi R^4}{2} = \frac{\pi \; (50)^4}{2} = 9.82 \; x \; 10^6 \; mm^4$$

$$J_{BC} = \frac{\pi R^4}{2} = \frac{\pi \; (25)^4}{2} = 6.13 \; x \; 10^6 \; mm^4$$

Free body diagram to determine reaction at A is shown in Figure 9.11 (b).

$$\sum T_x = 0; \quad \rightarrow \; -T_A + \; 10 - 4 = 0 \qquad \therefore T_A = 6 \; KNm$$

For part AB, refer to Figure 9.11 (c),

$$\sum T_x = 0; \quad \rightarrow \; -6 + \; T_{AB} = 0 \qquad \therefore T_{AB} = 6 \; KNm$$

For part AB, refer to Figure 9.11 (d),

$$\sum T_x = 0; \quad \rightarrow \; -6 \; + \; 10 \; + \; T_{BC} = 0 \qquad \therefore T_{BC} = -4 \; KNm$$

Calculate maximum shear stress

$$\tau_{max_{AB}} = \left(\frac{TR}{J}\right)_{AB} = \frac{(6\ x\ 10^{6})(50)}{9.82\ x\ 10^{6}} = 30\ \frac{N}{mm}^{2}$$

$$\tau_{max_{BC}} = \left(\frac{TR}{J}\right)_{BC} = \frac{(4\ x\ 10^{6})(25)}{6.13\ x\ 10^{5}} = 130\ \frac{N}{mm}^{2}$$

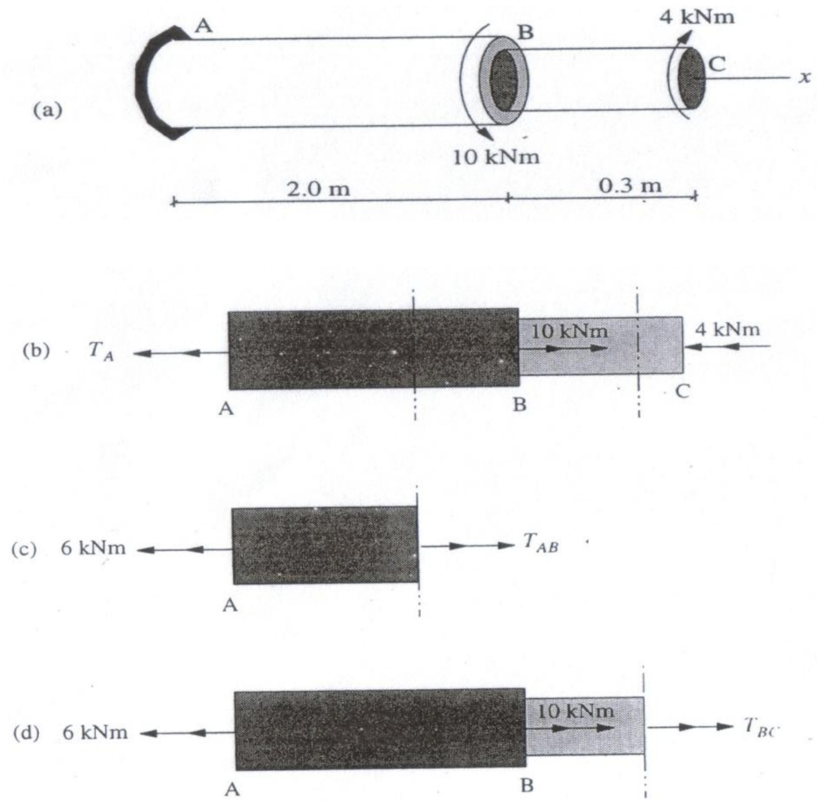


Figure 7.11: Torsion in combined bar

Maximum shear stress occurred in member BC. Twisting angle at C is

$$\emptyset_{AC} = \sum\frac{TL}{JG} = \left(\frac{TL}{JG}\right)_{AB} + \left(\frac{TL}{JG}\right)_{BC}$$

$$= \frac{(6\ x\ 10^{6})(2000)}{(9.82\ x\ 10^{6})(3\ x\ 10^{4})} + \frac{(-4\ x\ 10^{6})(300)}{(6.13\ x\ 10^{5})(8\ x\ 10^{4})}$$

$$\emptyset_{CA} = 0.0163\ rad$$ (rotating angle at C)

Example 7.3

Determine twisting angle at A for combined bar subjected to torsion as shown in Figure 9.12. Given $G = 80\ GPa = 80\ x\ \mathbf{10^9}\ \frac{N^2}{m}$.

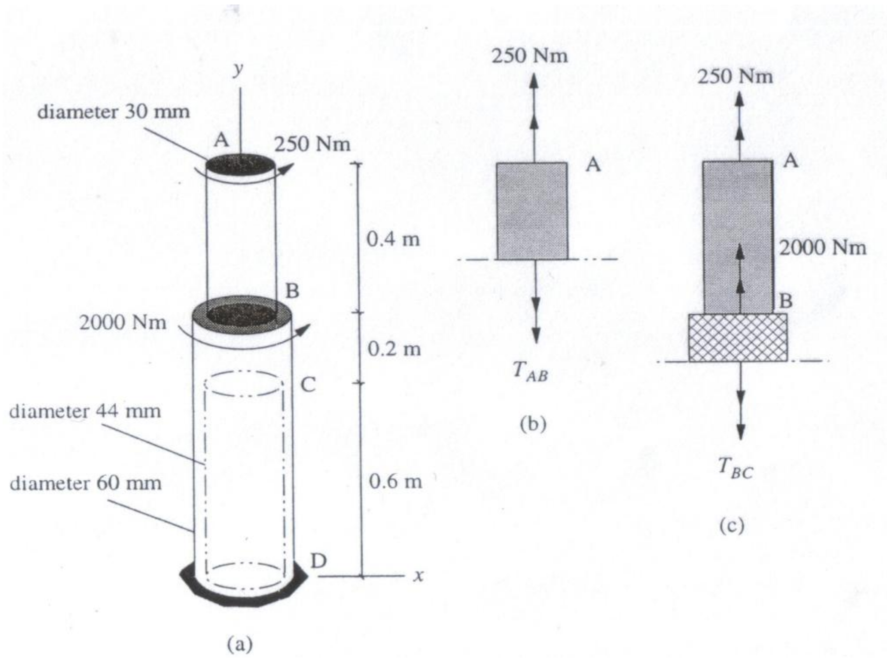


Figure 7.12: Torsion in combined bar

Solution

Free body diagram part AB, refer to Figure 9.12 (b),

$$\sum T_y = 0; \rightarrow 250 - \ T_{AB} = 0 \quad \therefore\ T_{AB} = 250\ Nm$$

Free body diagram part BC, refer to Figure 9.12 (c),

$$\sum T_y = 0; \rightarrow 250 + 2000 - \ T_{BC} = 0 \quad \therefore\ T_{BC} = 2250\ Nm$$

Due to no torsion at C,

$$T_{CD} = T_{BC} = 2250\ Nm$$

$$J_{AB} = \frac{\pi R^4}{2} = \frac{\pi (0.015)^4}{2} = 0.0795 \ x \ 10^{-6} \ m^4$$

$$J_{BC} = \frac{\pi R^4}{2} = \frac{\pi (0.03)^4}{2} = 1.272 \ x \ 10^{-6} \ m^4$$

$$J_{CD} = \frac{\pi (R_{ext}^4 - R_{int}^4)}{2} = \frac{\pi (0.03^4 - 0.022^4)}{2} = 0.904 \ x \ 10^{-6} m^4$$

$$\emptyset_A = \sum \frac{TL}{JG} = \left(\frac{TL}{JG}\right)_{AB} + \left(\frac{TL}{JG}\right)_{BC} + \left(\frac{TL}{JG}\right)_{CD}$$

$$\emptyset_A = \frac{250 \ x \ 0.4}{(0.0795 \ x \ 10^{-6})(80 \ x \ 10^9)} + \frac{2250 \ x \ 0.2}{(1.272 \ x \ 10^{-6})(80 \ x \ 10^9)} + \frac{2250 \ x \ 0.6}{(0.904 \ x \ 10^{-6})(80 \ x \ 10^9)}$$

$$\emptyset_A = 0.0388 \ rad$$

EXERCISE

Q7.1 Figure Q7.1 showed cross sectional of a bar subjected to torsion T . Prove that maximum shear stress produced in a rod is given as equation

$$\tau_{max} = \frac{TR}{J}$$

Then, by derivation formula J for respective cross section, prove that maximum shear stress can be written as

$$\tau_{max} = \frac{2TR}{\pi (R^4 - r^4)}$$

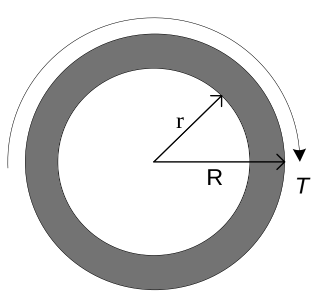


Figure Q7.1

Q7.2 Two rods are attached to each other and rigidly connected to its end as shown in Figure S7.2. Rod 1 is hollow tube and rod 2 is solid rod. Properties of both rods are given in Table 7.1.

Table 7.1

	Rod 1	**Rod 2**
τ_{max}	$50\ \frac{N}{mm}^{2}$	$40\ \frac{N}{mm}^{2}$
G	$80\ \frac{kN}{mm}^{2}$	$60\ \frac{kN}{mm}^{2}$
L	$800\ mm$	$600\ mm$
r_{int}	$20\ mm$	-

Determine external radius rod 1 and rod 2 so that this system is safe to support torsion of 1.4 kNm .

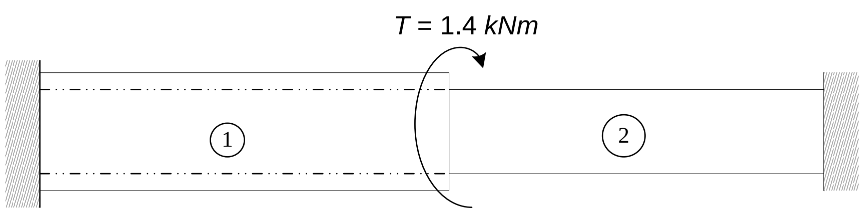


Figure Q7.2