Chapter 1

Stress and Strain

This chapter will discuss about the concept of stresses and strain created in various members and connection by the loads applied to a structure. The students also will learn an important aspect of the analysis and design of structures relates to the deformation caused by the loads applied to the structures. The mechanical properties of the selected materials also will be discussed with simple stress-strain diagram for a specific material

After successfully completing this chapter the student should be able to:

- defined the relationship between stress and strain
- analyse the stress and strain using related equations
- determine and analyse the deformation of a rod of uniform or variable cross section under one or several load
- determine the principal stress using equation and Mohr's circle method

1.0 Types and system of force

(i) Normal force

In geometry the word "normal" means perpendicular. Therefore, normal force can be defined as a force perpendicular to the plane or surface where an object is resting or moving. The force may be acting as a tension force (pull) or compression force (push). The SI unit is newton or N

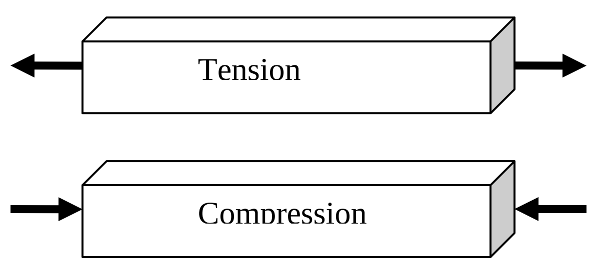


Figure 1.1: Tension and compression force

(ii) Shear force

Shear force can be defined as a force that attempts to cause the internal structure of a material to slide against itself. The force acting in a direction parallel to a surface of a body. Shear force also often result in shear strain. The SI unit of torque is Newton or N.

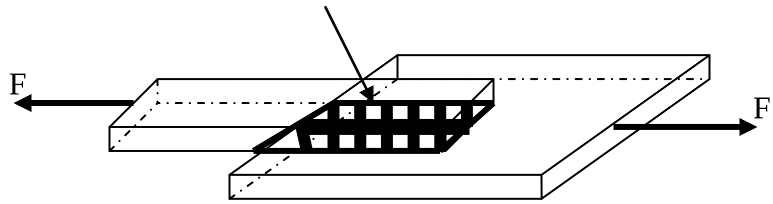


Figure 1.2: Shear force

(iii) Torque or Torsion
Torque is the tendency of a force to rotate an object about an axis. A torque can be though of as a twist ato an object. Mathematically, torque is defined as the product of force and the lever-arm distance, which tends to produce rotation. Torque is calculated by multiplying force and distance. The SI units of torque are Newton-meter or Nm.

1.1 Stress

Stress is defined as force per unit area. It has the same units as pressure, and in fact pressure is one special variety of stress. However, stress is a much more complex quantity than pressure because it varies both with direction and with the surface it acts on. Basically stress can be divided into three types:

(i) normal stress
(ii) bearing stress
(iii) shear stress

1.1.1 Normal stress

Normal stress is a stress that acts perpendicular to a surface. It is can be considered if the applied force is perpendicular to the plane of the cross sectional area under consideration. It is also can be either compression or tension. Compression stress is considered as a stress that causes an object shortening. Meanwhile tension stress is a stress that acts to lengthen an object. The stress in an axially loaded bar is:

$$\boxed{\sigma = \frac{P}{A}}$$

Stress is positive in tension ($P>0$ means $\sigma>0$), and negative in compression ($P<0$). English units: psi (pounds per square inch), or ksi (kilopounds per square inch). S.I. units: Pa (Pascal, N/m^2), or usually MPa (megapascal, 1 Mpa = 1,000,000 Pa).

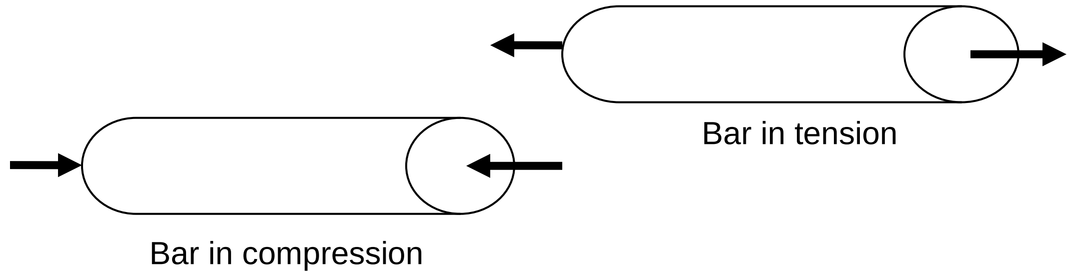
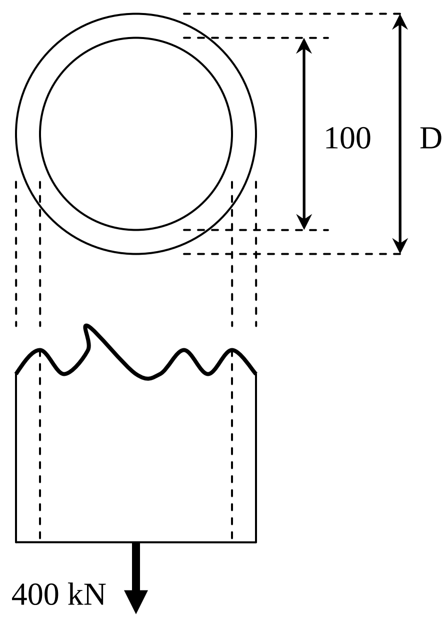


Figure 1.3: Normal stress in tension and compression

Example 1.1
A hollow steel tube with an inside diameter of 100 mm must carry a tensile load of 400 kN. Determine the outside diameter of the tube if the stress is limited to 120 MN/m^2.

Solution

Given:



P = 400 kN = 400 000 N
σ = 120 MPa

A= ¼ πD^2 – ¼ π(100^2)

A = ¼ π (D^2 – 10 000)

Thus,

400 000 = 120 [¼ π (D^2 – 10 000)]
400 000 = 30πD^2 – 30 000π

$$D^2 = \frac{400000 + 300000\pi}{30\pi}$$

D = 119.35 mm *answer*

Example 1.2
A homogeneous 800 kg bar AB is supported at either end by a cable as shown in Figure E1.2. Calculate the smallest area of each cable if the stress is not to exceed 90 MPa in bronze and 120 MPa in steel.

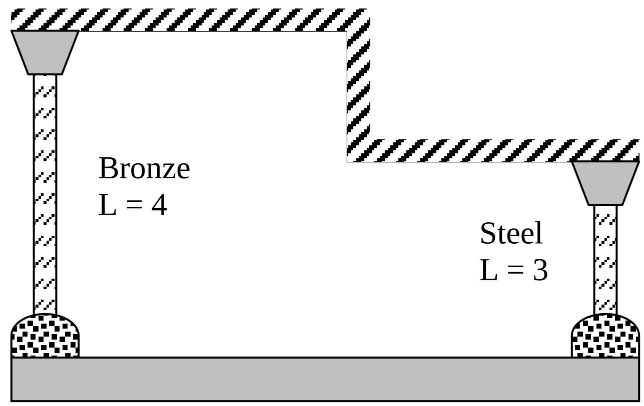
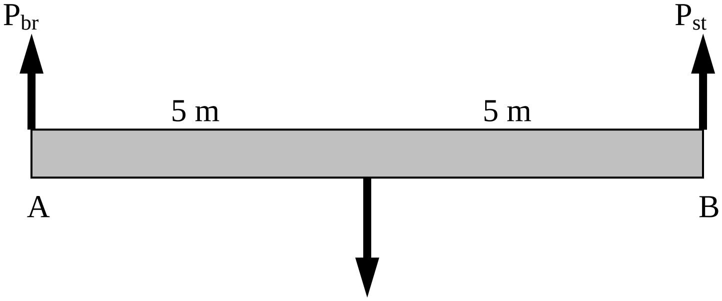


Figure E1.2

<u>Solution</u>



By symmetry:
$P_{br} = P_{st} = ½ (7848)$

$P_{br} = P_{st} = 3924$ N

For bronze cable:

$P_{br} = \sigma_{br}A_{br}$

$3924 = 90\ A_{br}$

$A_{br} = 43.6$ mm^2 answer

For steel cable:

$P_{st} = \sigma_{st}A_{st}$

$3924 = 120A_{st}$

$A_{st} = 32.7$ mm^2 *answer*

Example 1.3

An aluminum rod is rigidly attached between a steel rod and a bronze rod as shown in Figure E1.3. Axial loads are applied at the positions indicated. Find the maximum value of P that will not exceed a stress in steel of 140 MPa, in aluminum of 90 MPa, or in bronze of 100 MPa.

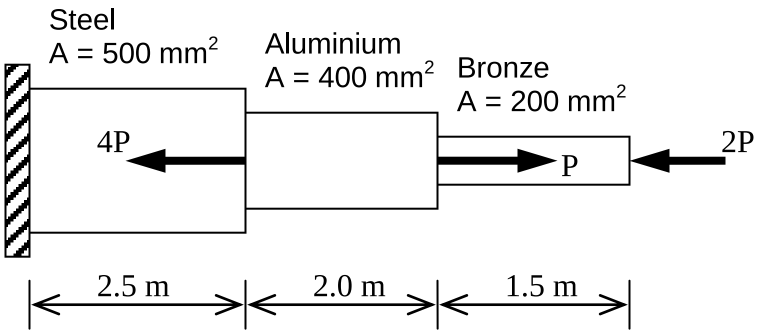
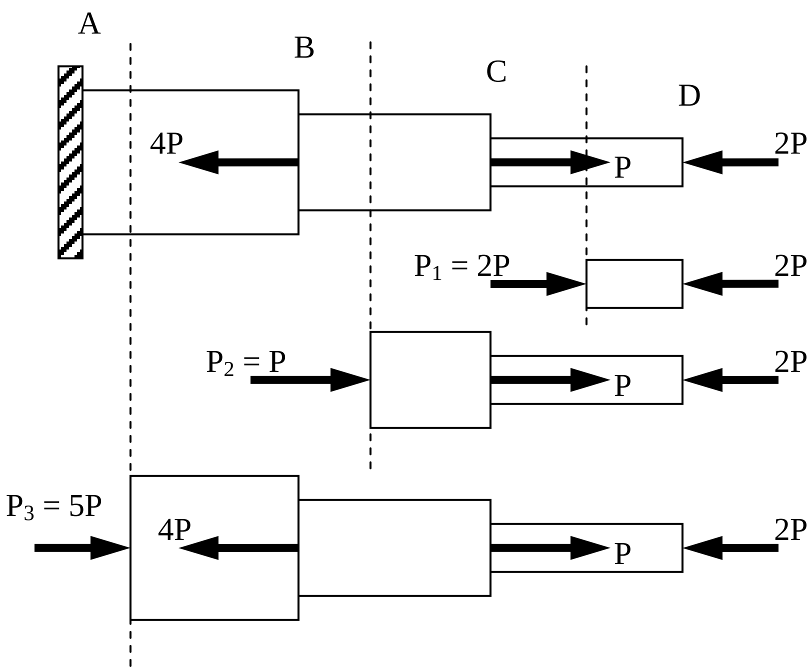


Figure E1.3

__Solution__



Note: The all loads are under compression (-ve)

Expressing that each of the free bodies is in equilibrium, therefore

$P_1 = -2P$
$P_2 = -P$
$P_3 = -5P$

For bronze:

$\sigma_{br} A_{br} = 2P$

$100(200) = 2P$

$P = 10\ 000$ N

For aluminum:

$\sigma_{al} A_{al} = P$

$90(400) = P$

$P = 36\ 000$ N

For Steel:

$\sigma_{st} A_{st} = 5P$

$140\ (500) = 5P$

$P = 14\ 000$ N

For safe value of P, use the smallest above. Thus,

$P = 10\ 000$ N $= 10$ kN *answer*

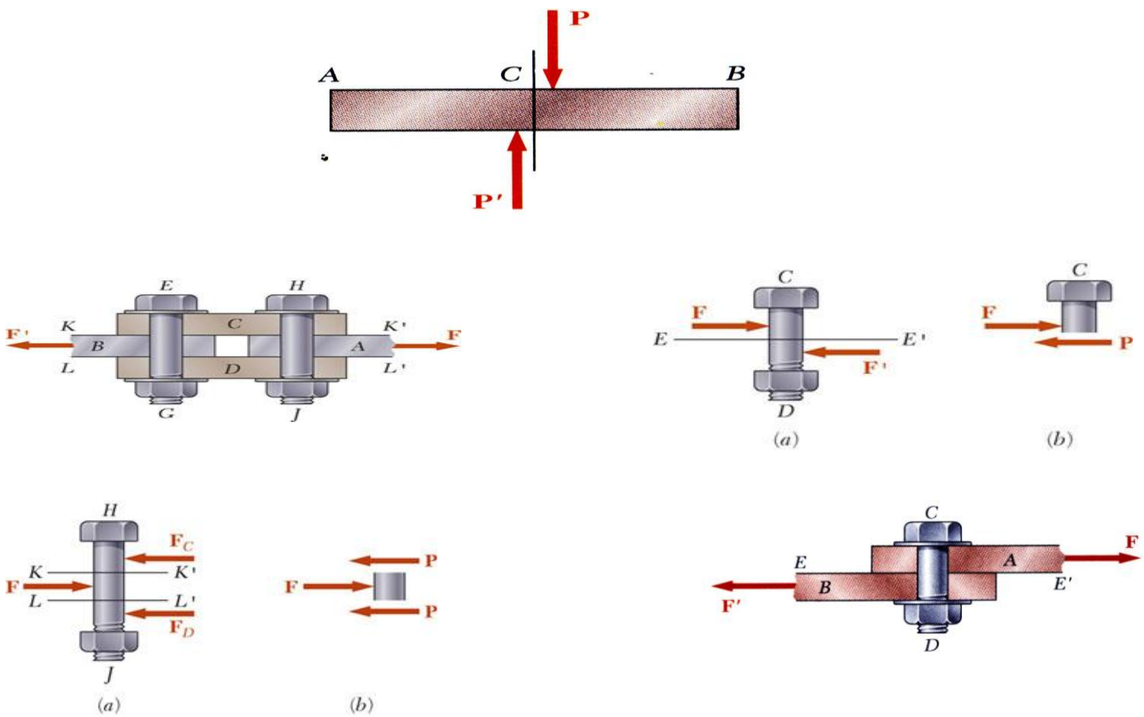
1.1.2 Shear stress

Shear stress is a stress that acts parallel to a surface. It can cause one object to slide over another. It also tends to deform originally rectangular objects into parallelograms. Shearing stress is also known as tangential stress. The most general definition is that shear acts to change the angles in an object.

$$\boxed{\tau = \frac{P}{A} = \frac{P}{hb}}$$

Where P = applied tensile force
A = area of the shearing plane between the two bars

The unit for shearing stress is same with the normal stress. The example of shearing stress are shown in Figure 1.5



$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{2A}$$

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A}$$

Figure 1.5: Shear stress

Example 1.4
Two wooden members of uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that P = 11 kN, determine the normal and shearing stresses in the glued splice.

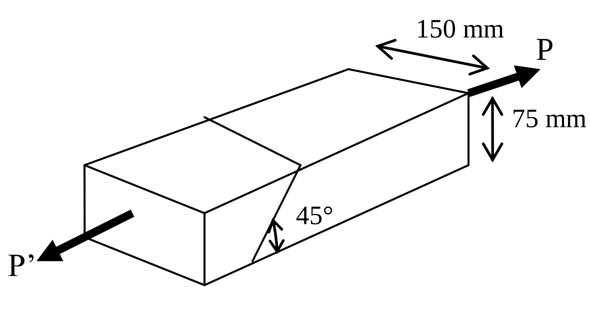


Figure E1.4

Solution:

$\theta = 90 - 45 = 45°$

P = 11 kN = 11 000 N

$A_o = (150)(75) = 11.25 \times 10^3\ mm^2 \Rightarrow 11.25 \times 10^{-3} m^2$

(i) Normal stress

$$\sigma = \frac{P\cos^2\theta}{A_o} = \frac{(11\times10^3)\cos^2 45^o}{11.25\times10^{-3}}$$

$\sigma = 489kPa$ *answer*

(ii) Shear stress

$$\tau = \frac{P\sin 2\theta}{2A_o} = \frac{(11\times10^3)\sin 90^o}{2(11.25\times10^{-3})}$$

P = 489 kPa *answer*

Example 1.5

The joint is fastened using two bolts as shown in figure below. Determine the required diameter of the bolts if allowable shear stress for the bolts is τ_{allow} = 110 Mpa

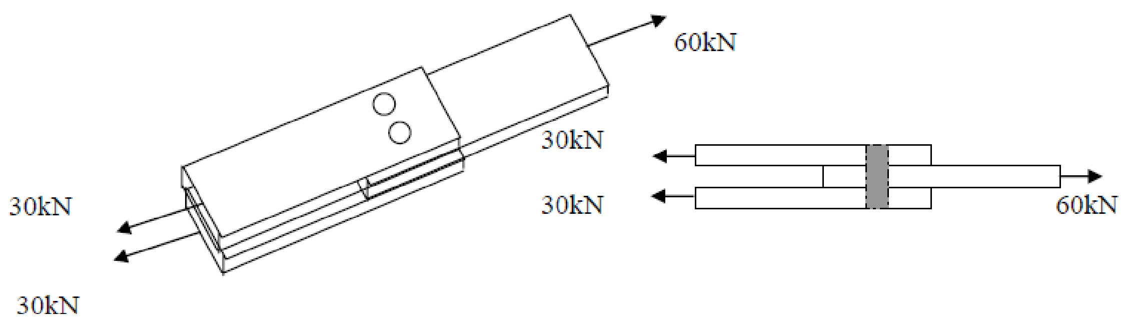


Figure E1.5

Solution

The figure above is double shear connection, therefore $\tau = \dfrac{P}{2A}$

However, the joint fastened using two bolts, so $\tau = \dfrac{P}{2(2A)} = \dfrac{P}{4A}$

$$A = \frac{\pi d^2}{4}$$

$$110 \times 10^6 = \frac{60 \times 10^3}{4\left(\dfrac{\pi d^2}{4}\right)}$$

d = 0.013 m

Example 1.6
A load P is applied to a steel rod supported as shown by an aluminium plate into which a 12 mm diameter hole has been drilled. Knowing that the shearing stress must not exceed 80 MPa in the stel rod and 70 Mpa in the aluminium plate, determine the largest load P that can be applied to the rod.

Solution

For the steel rod

$A_1 = \pi d_1 t_1 = \pi\,(0.012)(0.010)$

$= 376.99 \times 10^{-6}\ m^2$

$$\tau_1 = \frac{P_1}{A_1}$$

$P_1 = (180 \times 10^6)(376.99 \times 10^{-6}) = 67.86 \times 10^3$ N

For the aluminium plate,

$A_2 = \pi d_2 t_2 = \pi\,(0.040)(0.008)$

$= 1.01 \times 10^{-3}\ m^2$

$$\tau_2 = \frac{P_2}{A_2}$$

$P_2 = (70 \times 10^6)(1.01 \times 10^{-3})$

$= 70.372 \times 10^3$ N

The limiting value for the load P is the smaller of P_1 and P_2.

P = 67.89 kN

1.1.3 Bearing Stress

Bearing stress is a type of normal stress but it involves the interaction of two surfaces. The bearing stress is the pressure experience by the second surface due to the action from the first surface. Example: the pressure between bolt and plate at a joint.

$$\boxed{\sigma = \frac{P}{A} = \frac{P}{tD}}$$

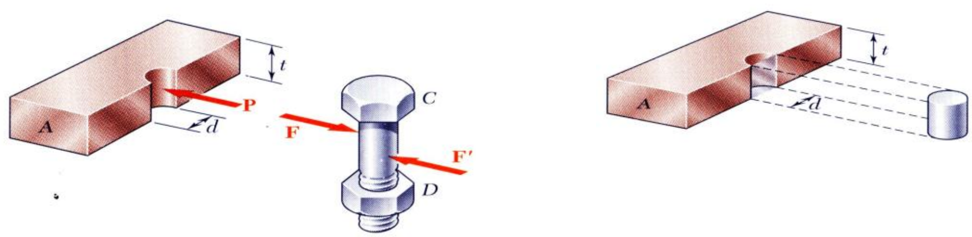


Figure 1.6: Bearing stress

Example 1.7

In Figure E1.7, assume that a 20-mm-diameter rivet joins the plates that are each 110 mm wide. The allowable stresses are 120 MPa for bearing in the plate material and 60 MPa for shearing of rivet. Determine (a) the minimum thickness of each plate; and (b) the largest average tensile stress in the plates.

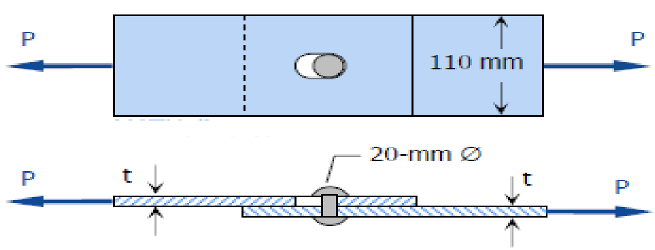


Figure E1.7

<u>Solution:</u>

Part (a):
From shearing of rivet:

$$P = \tau A_{rivets}$$
$$P = 60\left[\frac{1}{4}\pi\left(20^2\right)\right]$$
$$P = 6000\pi$$

From bearing of plate material:

$$P = \sigma_b A_b$$
$$6000\pi = 120(20t)$$
$$t = 7.85mm$$

Part (b): Largest average tensile stress in the plate:

$$P = \sigma A$$
$$6000\pi = \sigma[7.85(110 - 20)]$$
$$\sigma = 26.67MPa$$
$$P = \sigma A$$

1.2 Strain

Strain is a measure of deformation of a body which undergoes elongation, contraction or twisted through a certain angle. Generally, strain can be classified into two types namely:

(i) normal strain (ε)
(ii) shear strain (γ)

1.2.1 Normal strain

Normal strain (ε) is the deformation of a body which involved elongation or contraction. When a bar of length L and cross-sectional area A is subjected to axial tensile force P through the cross-section's centroid, the bar elongates. The change in length divided by the initial length is the bar's engineering strain. The symbol for strain is ε (epsilon). The strain in an axially loaded bar is:

$$\boxed{\varepsilon = \frac{\delta}{L}}$$

Strain is positive in tension and negative in compression. Strain is a fractional change in length (it is dimensionless). Due to the strain is much smaller than 1, it is typically given as a percentage: e.g., = 0.003 = 0.3%.

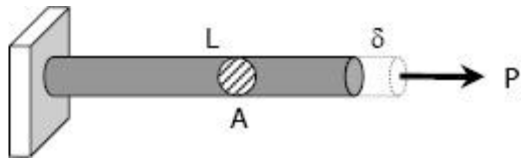


Figure 1.7: Normal strain

1.2.2 Shear strain

Shear strain is a strain which involved a shear deformation i.e. body twist due to torsion and a distorted cuboid as shown in Figure 1.8. Strain changes the angles of an object and shear causes lines to rotate.

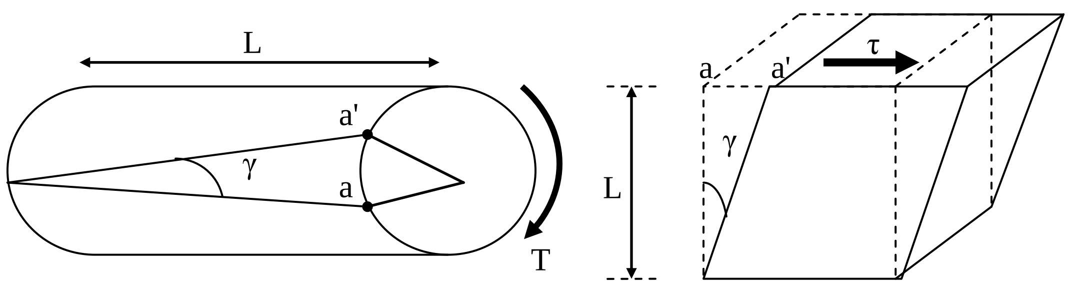


Figure 1.8 Shear strain due to twisting moment (T) and shear stress (τ)

Shear strain $$\boxed{\gamma = \frac{aa'}{L}}$$ but it is considered small in practice

The relationship between the shear strain, shear stress and the modulus of rigidity is as follows:

$$\boxed{\gamma = \frac{\tau}{G}}$$

Where τ = shear stress
γ = shear strain in radians
G = modulus of rigidity

1.3 Normal stress and strain relationship

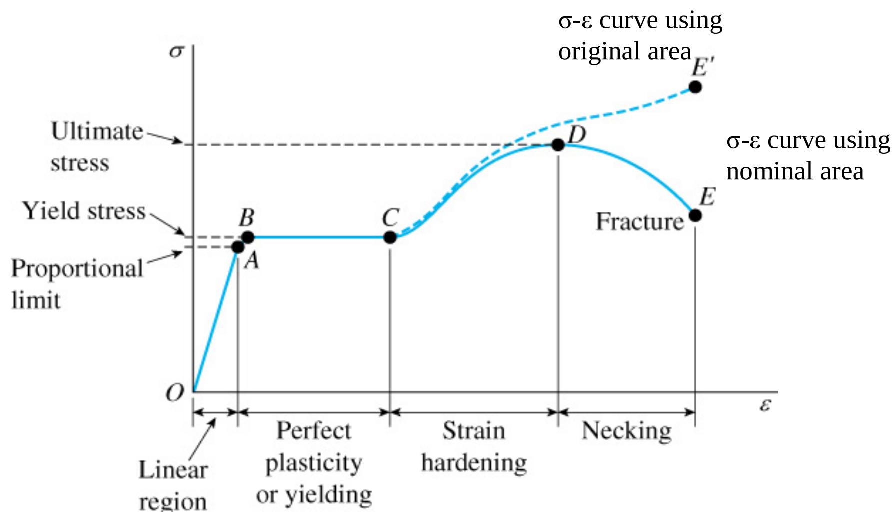


Figure 1.9: The stress-strain relationship

The stress-strain relationship of a material usually can be obtained from tensile or compression test on a specimen of the material. Figure 1.9 shows the stress-strain behavior which indicates how the material deforms on the application of load. The normal stress for the material is computed by dividing the load (P) by the original cross-sectional area (A). Stress-strain diagrams of various materials vary widely, and different tensile tests conducted on the same material may yield different result, depending upon the temperature of the specimen and the speed loading.

Proportional Limit (Hooke's Law)
From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called Hooke's Law that within the proportional limit, the stress is directly proportional to strain. The constant of proportionality k is called the Modulus of Elasticity E or Young's Modulus and is equal to the slope of the stress-strain diagram from O to P.

Elastic Limit
The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may e developed such that there is no permanent or residual deformation when the load is entirely removed. However, in practice this point is very difficult to determine because very close to proportional limit point.

Yield Point

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load. The material is said to undergo plastic deformation.

Strain hardening

Point C to D is called as strain hardening region whereas the curve rises gradually until it flatten at D. The stress which correspond to point D is called ultimate strength/stress

Ultimate Strength/Stress

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

Rapture Strength (Fracture)

Rapture strength is the strength of the material at rupture. This is also known as the breaking strength (final point).

1.3.1 Offset Method

Beside steel, other materials such as aluminium, glass, brass and zinc, constant yielding will not occur beyond the elastic range. This metal often does not have a well defined yield point. Therefore, the standard practice to define yield strength for this metal is graphical procedure called the offset method. Normally a 0.2% (0.002 mm/mm) is chosen, and from this point on the strain (ε) axis, a line parallel to the initial straight-line portion of the stress-strain diagram is drawn (Figure 1.10). The point where this line intersects the curves defines the yield strength.

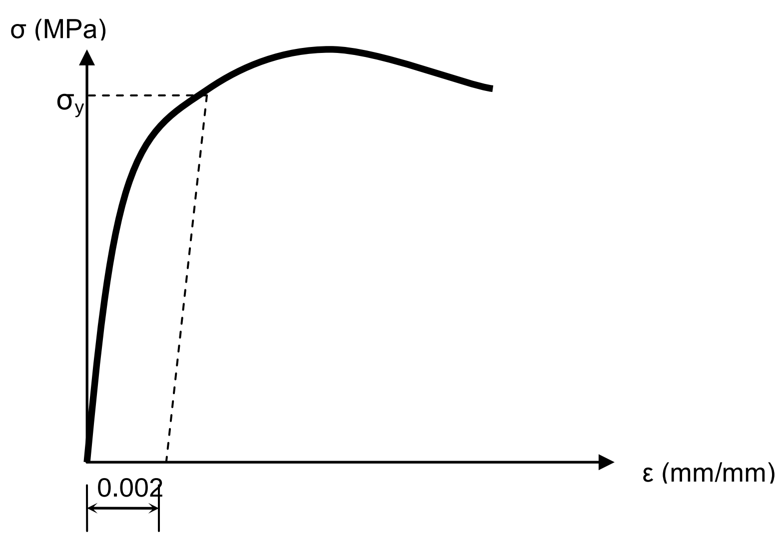


Figure 1.10: Determination of yield strength using offset method

Example 1.8

A rod with the diameter 5 mm and length 100 mm is stressed slowly with the load up to failure. The result for this test is shown in Table E1.8. Draw the stress-strain curve and determine

(a) modulus of elasticity

(b) yield stress

(c) stress maximum

Table E1.8

Force P (N)	Elongation δ (mm)
1100	0.0625
2200	0.0125
3300	0.1875
3740	0.2375
4180	0.2875
4620	0.4275
4840	0.5300
5060	0.7625
5280	0.8900
5060	1.0250
4840	1.1525

Solution

Force P (N)	Elongation δ (mm)	Strain ε	Stress σ (N/mm²)
1100	0.0625	0.00063	56.04
2200	0.0125	0.00013	112.07
3300	0.1875	0.00188	168.11
3740	0.2375	0.00238	190.52
4180	0.2875	0.00288	212.94
4620	0.4275	0.00428	235.35
4840	0.5300	0.00530	246.56
5060	0.7625	0.00763	257.77
5280	0.8900	0.00890	268.98
5060	1.0250	0.01025	257.77
4840	1.1525	0.01153	246.56

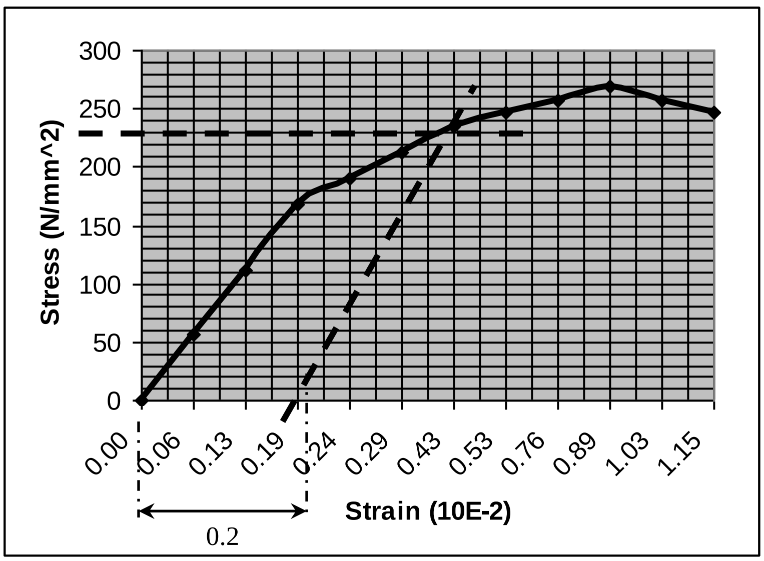


Figure E1.8

From the graph;

(a) $E = 112.07/0.00125 = 89\,600$ N/mm^2

(b) $\sigma_y = 230$ N/mm^2

(c) $\sigma_{max} = 270$ N/mm^2

1.4 Hooke's Law

Stiffness; Modulus Young

Stiffness is a material's ability to resist deformation. The stiffness of a material is defined through Hooke's Law:

$$\boxed{\sigma = E\varepsilon}$$

where E is Young's Modulus (the modulus of elasticity**),** a material property. Values of E for different materials are obtained experimentally from stress-strain curves. Young's Modulus is the slope of the linear-elastic region of the stress-strain curve.

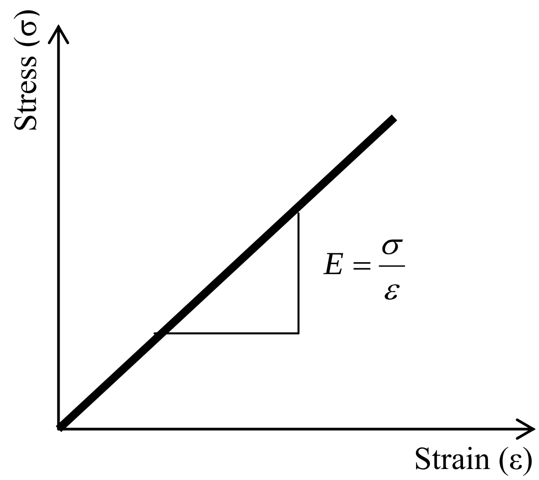


Figure 1.11: Stress-strain relationship at the linear-elastic region

E is generally large and given in either ksi (kilopounds per sq.inch) or Msi (megapounds per sq. inch = thousands of ksi), or in GPa (gigapascal).

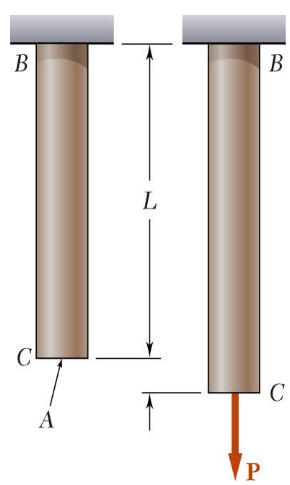


Figure 1.12: Deformation due to axial load

Consider a homogenous rod BC of length L and uniform cross section of area A subjected to a centric axial load P (Figure 1.12). If the resulting axial stress σ = P/A does not exceed the proportional limit of the material, the Hooke's law can be apply and write as follow:

$$\sigma = E\varepsilon$$

From which it follows that

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

The strain;

$$\varepsilon = \frac{\delta}{L}$$

So,

$$\delta = \varepsilon L$$

Therefore;

$$\boxed{\delta = \frac{PL}{AE}}$$

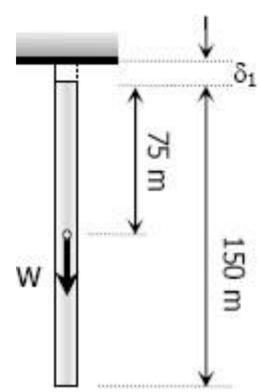
Example 1.9

A steel rod having a cross-sectional area of 300 mm^2 and a length of 150 m is suspended vertically from one end. It supports a tensile load of 20 kN at the lower end. If the unit mass of steel is 7850 kg/m^3 and E = 200 × 10^3 MN/m^2, find the total elongation of the rod.

Solution

Elongation due to its own weight:

$$\delta_1 = \frac{PL}{AE}$$



Where:

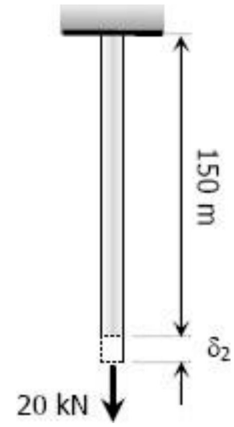
P = W = 7850x 300 (1/1000^2) x 150 x 9.81
P = 3465.3825 N
L = 75(1000) = 75 000 mm
A = 300 mm^2
E = 200 000 MPa

Thus,

$$\delta_1 = \frac{3465.3825(75000)}{300(200000)}$$
$$\delta_1 = 4.33mm$$

Elongation due to applied load:

$$\delta_2 = \frac{PL}{AE}$$



Where:
P = 20 kN = 20 000 N
L = 150 m = 150 000 mm
A = 300 mm^2
E = 200 000 MPa

Thus,

$$\delta_2 = \frac{20000(150000)}{300(200000)}$$
$$\delta_2 = 50mm$$

Total elongation:

$$\delta = \delta_1 + \delta_2$$
$$\delta = 4.33 + 50$$
$$\delta = 54.33mm$$

Example1.10
Determine the deformation of the steeel rod shown in Figure E1.10 under the given loads (E=200 GPa)

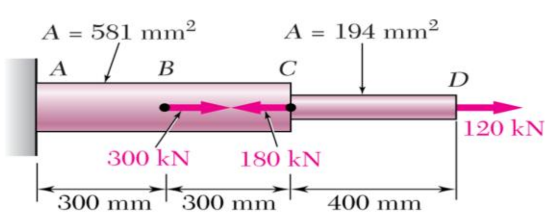
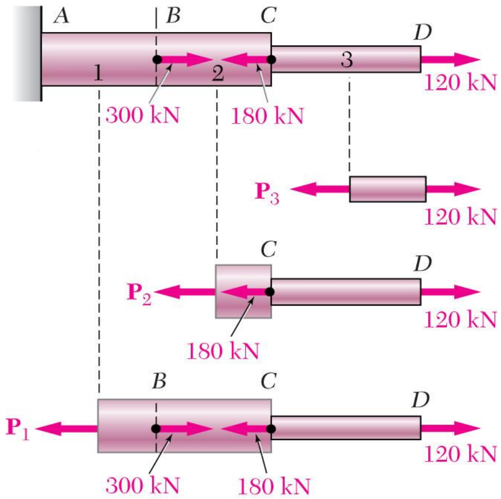


Figure E1.10



$L_1 = L_2 = 300$ mm
$A_1 = A_2 = 581$ mm^2 = 581×10^{-6} m^2

$$P_1 = 240 \times 10^3\,\text{N}$$
$$P_2 = -60 \times 10^3\,\text{N}$$
$$P_3 = 120 \times 10^3\,\text{N}$$

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$
$$= \frac{1}{200 \times 10^9}\left[\frac{(240 \times 10^3)0.3}{581 \times 10^{-6}} + \frac{(-60 \times 10^3)0.3}{581 \times 10^{-6}} + \frac{(120 \times 10^3)0.4}{194 \times 10^{-6}}\right]$$
$$= 1.73 \times 10^{-3}\,\text{m}$$

1.5 Poisson ratio

Poisson's ratio is the ratio of lateral contraction strain to longitudinal extension strain in the direction of stretching force. Tensile deformation is considered positive and compressive deformation is considered negative. The definition of Poisson's ratio contains a minus sign so that normal materials have a positive ratio. Poisson's ratio, also called Poisson ratio or the Poisson coefficient. Poisson's ratio is a materials property.

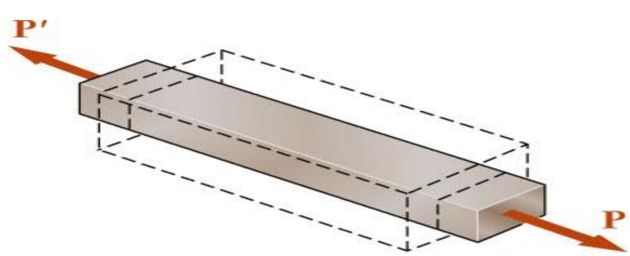
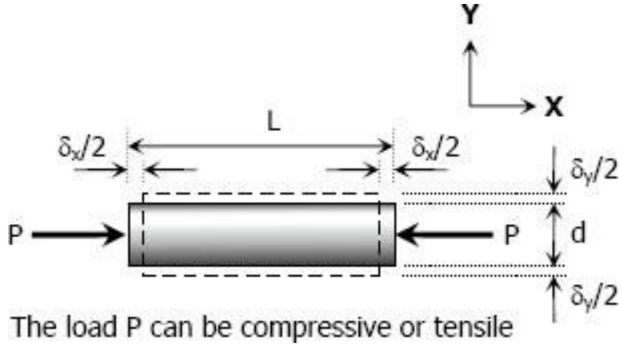


Figure 1.9: Lateral and longitudinal strain is same in all direction

$$\boxed{\upsilon = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}}$$

Example 1.11
A solid cylinder of diameter d carries an axial load P. Show that its change in diameter is 4Pv / πEd



Solution

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E}$$

$$\frac{\delta_y}{d} = -\nu\frac{-P}{AE}$$

$$\delta_y = \frac{Pd}{\frac{1}{4}\pi d^2 E}$$

$$\delta_y = \frac{4P\upsilon}{\pi Ed}$$

1.6 Working stress, permissible stress and temperature stress

1.6.1 Temperature Stress

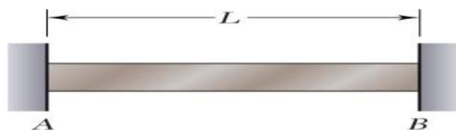
An object will expand when heated and contract when the temperature drops. This phenomenon is very important because if these movements are prevented, then internal stress and strain will be developed within the body of the structural member and the effect can be very disastrous. Since this is the effect of temperature on the member then the corresponding stress and strain are called temperature stress and temperature strain. For this reason, most civil engineering structure is provided with expansion joint to allow for free expansion and contraction of the member.

The variation of the length due to temperature change depends upon its coefficient of linear expansion or contraction α where α is the change in length for a unit change of temperature per unit original length.

1.6.2 Superposition Method

This method is applied for indeterminate problem where the reactions at the support are more than what is required to maintain its equilibrium. In this method, one of the support is released and let it elongate freely as it undergoes the temperature change ΔT.

Step 1 Consider a rod AB is placed between two fixed supports. Assuming there is no temperature stress or strain in this initial condition.

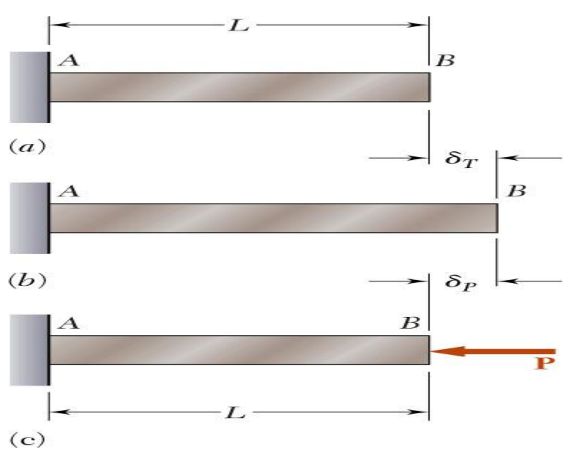


Step 2 Released the support B and let it elongate freely as it undergoes the temperature change ΔT. The corresponding elongation (δ_T) is:

$$\delta_T = \alpha(\Delta T)L$$

Step 3 Applying to end B the force (P) representing the redundant reaction and we obtain a second deformation (δ_P):

$$\delta_P = \frac{PL}{AE}$$



The total deformation must be zero:

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

From which, we conclude that

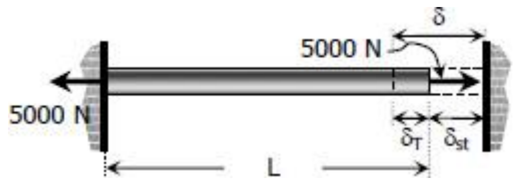
$$P = -AE\alpha(\Delta T)$$

An the stress in the rod due to the temperature change is

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Example 1.12

A steel rod is stretched between two rigid walls and carries a tensile load of 5000 N at 20°C. If the allowable stress is not to exceed 130 MPa at -20°C, what is the minimum diameter of the rod? Assume α = 11.7 µm/(m·°C) and E = 200 GPa.



Solution

$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

Therefore:

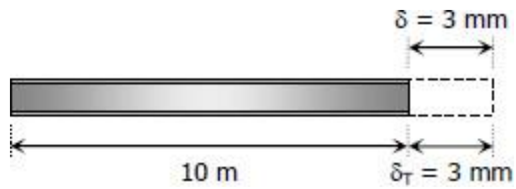
$$130 = (11.7\times10^{-6})(200000)(40) + \frac{5000}{A}$$

$$A = \frac{5000}{36.4} = 137.36 mm^2$$

$$\frac{1}{4}\pi d^2 = 137.36$$

$$d = 13.22 mm$$

Example 1.13
Steel railroad reels 10 m long are laid with a clearance of 3 mm at a temperature of 15°C. At what temperature will the rails just touch? What stress would be induced in the rails at that temperature if there were no initial clearance? Assume α = 11.7 µm/(m·°C) and E = 200 Gpa.



<u>Solution</u>

Temperature at which δ_T = 3 mm:

$$\delta_T = \alpha L(\Delta T)$$

$$\delta_T = \alpha L(T_f - T_i)$$

$$3 = (11.7 \times 10^{-6})(10000)(T_f - 15)$$

$$T_f = 40.64°C$$

Required stress:

$$\delta = \delta_T$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T)$$

$$\sigma = \alpha E(T_f - T_i)$$

$$\sigma = (11.7 \times 10^{-6})(200000)(40.64 - 15)$$

$$\sigma = 60 MPa$$

Example 1.14

The rigid bar ABC in Figure E1.14 is pinned at B and attached to the two vertical rods. Initially, the bar is horizontal and the vertical rods are stress-free. Determine the stress in the aluminum rod if the temperature of the steel rod is decreased by 40°C. Neglect the weight of bar ABC.

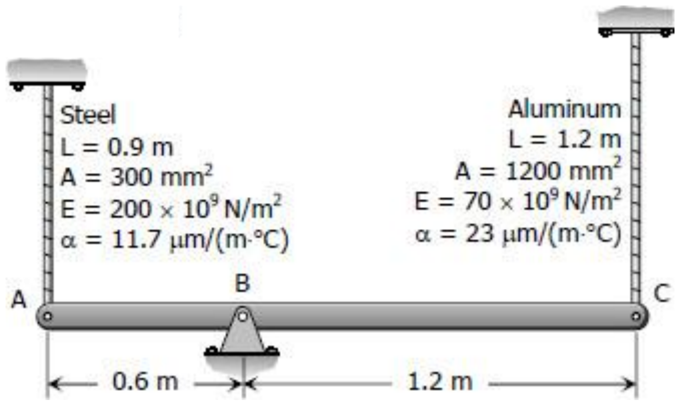


Figure E1.14

Solution

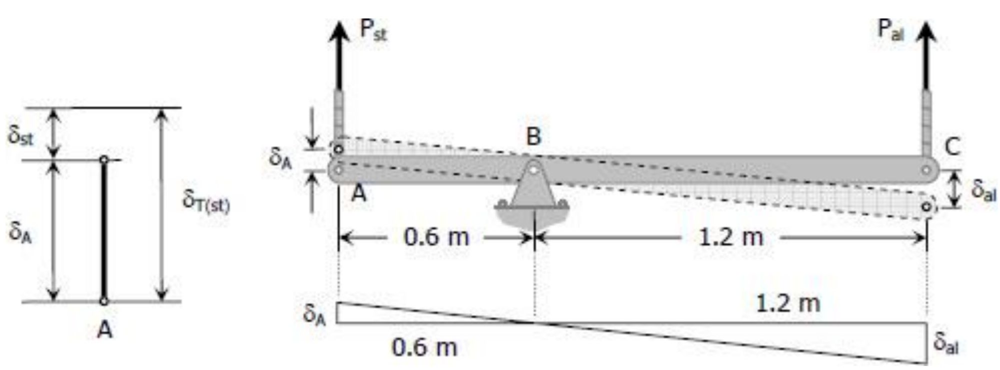
Contraction of steel rod, assuming complete freedom:

$$\delta_{T(st)} = \alpha L \Delta T$$

$$\delta_{T(st)} = (11.7 \times 10^{-6})(900)(40)$$

$$\delta_{T(st)} = 0.4212 mm$$

The steel rod cannot freely contract because of the resistance of aluminum rod. The movement of A (referred to as δ_A), therefore, is less than 0.4212 mm. In terms of aluminum, this movement is (by ratio and proportion):



$$\frac{\delta_A}{0.6} = \frac{\delta_{al}}{1.2}$$

$$\delta_A = 0.5\delta_{al}$$

$$\delta_{T(st)} = \delta_A + \delta_{st}$$

$$\delta_{T(st)} - \delta_{st} = \delta_A$$

$$\delta_{T(st)} - \delta_{st} = 0.5\delta_{al}$$

$$0.4212 - \left(\frac{PL}{AE}\right)_{st} = 0.5\left(\frac{PL}{AE}\right)_{al}$$

$$0.4212 - \frac{P_{st}(900)}{300(200000)} = 0.5\left[\frac{P_{al}(1200)}{1200(70000)}\right]$$

$28080 - P_{st} = 0.4762P_{al}$ ===➔ Equation 1

$$\sum M_B = 0$$

$$0.6P_{st} = 1.2P_{al}$$

$P_{st} = 2P_{al}$ ===➔ Equation 2

Equations (2) in (1)

$28080 - 2P_{al} = 0.4762P_{al}$

$P_{al} = 11340\ N$

$$\sigma_{al} = \frac{P_{al}}{A_{al}} = \frac{11340}{1200}$$

$\sigma_{al} = 9.45 MPa$ *answer*

1.7 Plane Stress and Mohr Circle

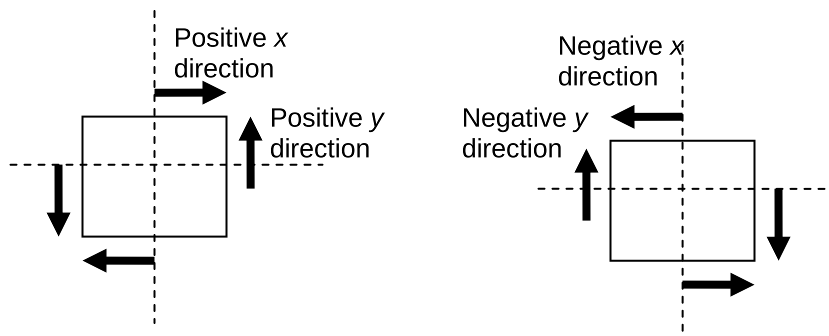
1.7.1 Sign Convention

Normal stress

(a) Compression stress (b) Tensile stress

Figure 1.14: Sign convention for axial stress

Refer to Figure 1.14, a tensile stress is considered to be positive normal stress, meanwhile, compressive stress is a negative stress

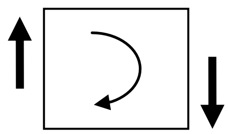


(a) Shear stress (+ve) (b) Shear stress (-ve)

Figure 1.15: Shear stress sign convention in application to plane stress equations

In application to plane stress equations (Figure 1.15), the sign convention will determined the sign of the shear stress that to be used in the plane stress equations. The shear stress is considered positive if:

(i) It acts on the positive x-surface in the positive y-direction
(ii) It acts on the positive y-surface in the positive x-direction



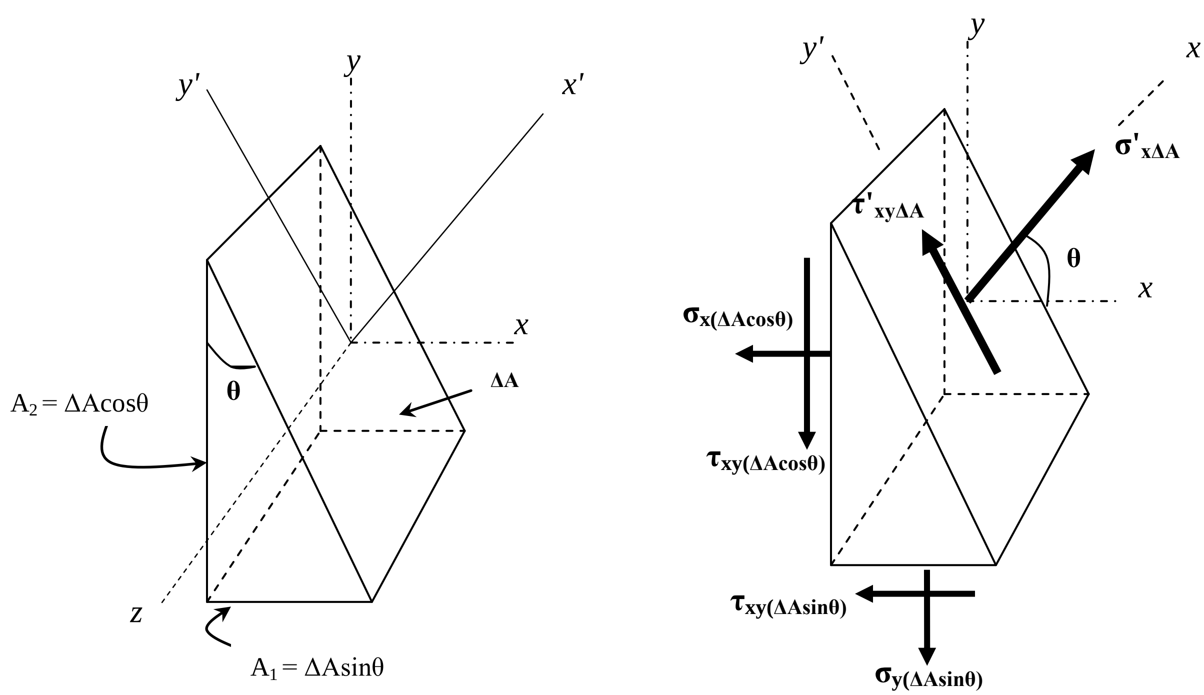
(a) Shear stress (+ve) (b) Shear stress (-ve)

Figure 1.16: Shear stress sign convention in application to Mohr's Circle

In application to Mohr's Circle (Figure 1.16), the sign convention will determine the coordinate of the point on the Mohr's Circle. The shear stress is considered positive (+ve) if it cause the element to rotate in clockwise direction. Meanwhile, the negative stress will be considered if the shear stress causes the element to rotate in counterclockwise.

1.7.2 Stress Analysis Using Equation and Mohr Diagram Method

1.7.2.1 Equation Method



$$\sin\theta = \frac{A_1}{\Delta A} \qquad\qquad \cos\theta = \frac{A_2}{\Delta A}$$

and

$$A_1 = \Delta A \sin\theta \qquad\qquad A_2 = \Delta A \cos\theta$$

$\sigma = \dfrac{P}{A}$ therefore, $P = \sigma A$

<u>For *x*-axis</u>

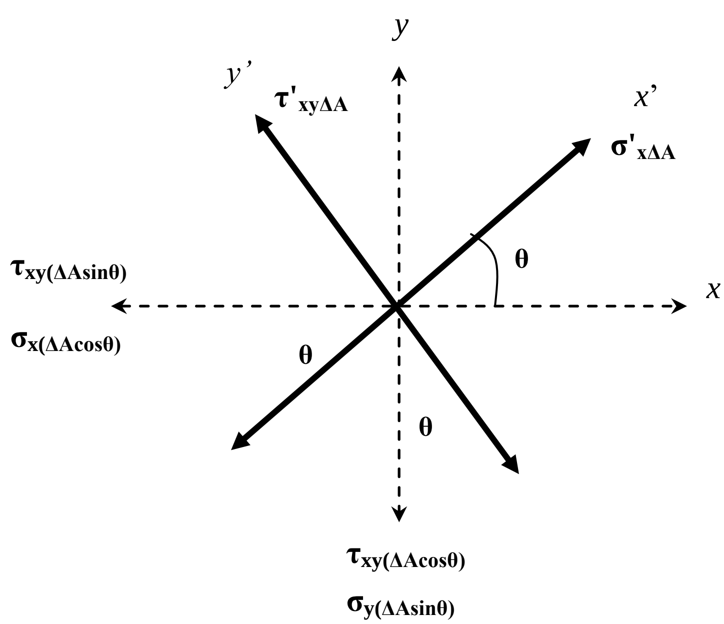
$P_x = \sigma_x A_2$ therefore $P_x = \sigma_x \Delta A \cos\theta$

$P_x = \tau_{xy} A_2$ therefore $P_x = \tau_{xy} \Delta A \cos\theta$

<u>For y-axis</u>

$P_y = \sigma_y A_1$ therefore $P_y = \sigma_y \Delta A \sin\theta$

$P_y = \tau_{xy} A_1$ therefore $P_y = \tau_{xy} \Delta A \sin\theta$



Using components in *x'* and *y'* axes, the equilibrium equations are:

$\sum F'_x = 0$ and $\sum F'_y = 0$

The equilibrium in *x'* axis ($\sum F'_x = 0$)

$\sigma'_x \Delta A - \sigma_x \Delta A \cos\theta\cos\theta - \tau_{xy}\Delta A \cos\theta\sin\theta - \sigma_y \Delta A \sin\theta\sin\theta - \tau_{xy}\Delta A \sin\theta\cos\theta = 0$

$\sigma'_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta = 0$ Equation 1.1

The equilibrium in *y'* axis ($\sum F'_y = 0$)

$\tau'_{xy}\Delta A + \sigma_x \Delta A \cos\theta\sin\theta - \tau_{xy}\Delta A \cos\theta\cos\theta - \sigma_y \Delta A \sin\theta x \cos\theta + \tau_{xy}\Delta A \sin\theta x \sin\theta = 0$

$\tau'_{xy} = (\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$ Equation 1.2

Using the trigonometric relationship

$Sin 2\theta = 2\sin\theta\cos\theta$ Equation 1.3

$Sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ Equation 1.4

$Cos 2\theta = 1 - 2\sin^2\theta$ Equation 1.5

$Cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ Equation 1.6

$Cos 2\theta = 2\cos^2\theta - 1$ Equation 1.7

Substitute Equations 1.3, 1.4 and 1.6 into Equation 1.1, becomes

$$\sigma'_x = \sigma_x\left[\frac{1+\cos 2\theta}{2}\right] + \sigma_y\left[\frac{1-\cos 2\theta}{2}\right] + \tau_{xy}\sin 2\theta$$

$$\sigma'_x = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta + \tau_{xy}\sin 2\theta$$ Equation 1.8

Substitute Equations 1.3, 1.4 and 1.6 into Equation 1.2, becomes

$$\tau'_{xy} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$ Equation 1.9

The expression for the normal stress σ'$_y$ is obtained by replacing θ in Equation 1.8 by the angle θ+90° that the y' axis form with x axis. Since cos (2θ+180°) = -cos2θ and sin(2θ+180°) = -sin2θ, so

$$\sigma_{y'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] - \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta - \tau_{xy}\sin 2\theta$$ Equation 1.10

The principal stress or σ_{max} and σ_{min} can be calculated by differentiate the Equation 1.8:

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_{x'}}{d\theta} = -\left[\frac{\sigma_x - \sigma_y}{2}\right]2\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

$$\frac{d\sigma_{x'}}{d\theta} = 0$$

Therefore;

$$-\left[\frac{\sigma_x - \sigma_y}{2}\right]2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\left[\frac{\sigma_x - \sigma_y}{2}\right]2\sin 2\theta = 2\tau_{xy}\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos\theta} = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \quad \Rightarrow \quad \tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

θ = θ_{P1} and θ_{P2} where θ_{P1} and θ_{P2} are 180° apart

Thus , $\tan 2\theta_{P1} = \tan 2\theta_{P2} = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$ Equation 1.11

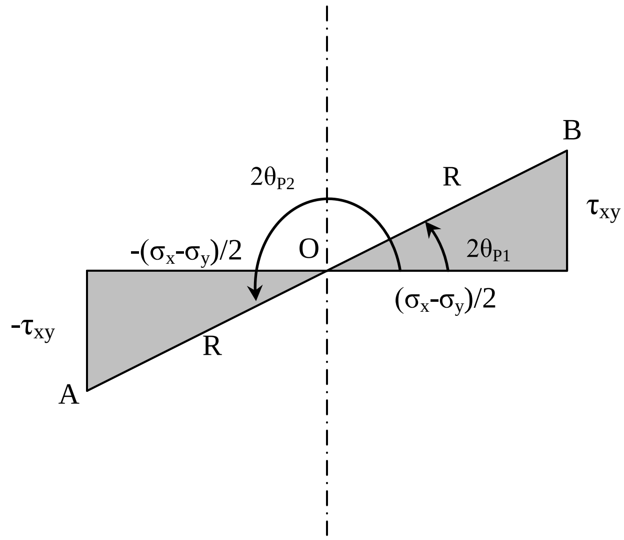


Figure 1.17: Location of $2\theta_{P1}$ and $2\theta_{P2}$

The length of OA and OB in Figure 1.17 is $= \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$

Thus from the trigonometry of the triangles, we have

$$\cos 2\theta_{P1} = \frac{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$ Equation 1.12 (a)

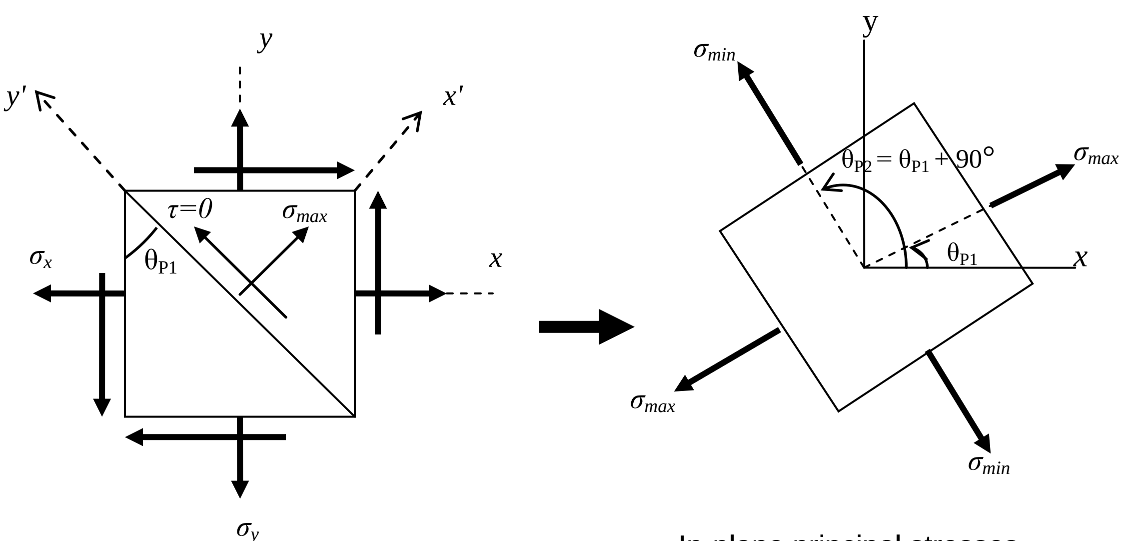
$$\sin 2\theta_{P1} = \frac{\tau_{xy}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$ Equation 1.12(b)

$$\cos 2\theta_{P2} = \frac{-\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$ Equation 1.12(c)

$$\sin 2\theta_{P2} = \frac{-\tau_{xy}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$ Equation 1.12(d)

Substituting Equation 1.12(a) to 1.12(d) into Equation 1.8, thus

$$\sigma_{\max,}, \sigma_{\min} = \left[\frac{\sigma_x + \sigma_y}{2}\right] \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$$ Equation 1.13



In-plane principal stresses

The maximum and minimum shear stresses can be determined by differentiating Equation 1.9.

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_{x'y'}}{d\theta} = 0$$

$$\frac{d\tau_{x'y'}}{d\theta} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)2\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

$$\frac{d\tau_{x'y'}}{d\theta} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)2\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

$$2\tau_{xy}\sin 2\theta = -\left(\frac{\sigma_x - \sigma_y}{2}\right)2\cos 2\theta$$

$$\tan 2\theta = \frac{-(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$2\theta = 2\theta_{S1}$ and $2\theta_{S2} = 2\theta_{S1} + 180°$

Where $\tan 2\theta_{S1} = \dfrac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$ and $\tan 2\theta_{S2} = \dfrac{(\sigma_x - \sigma_y)}{-2\tau_{xy}}$ Equation 1.14

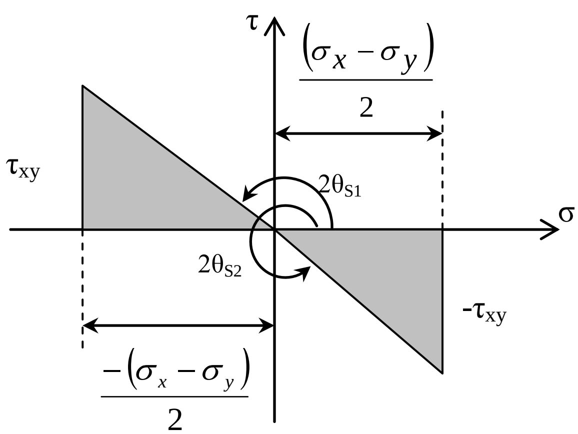


Figure 1.18: Location of $2\theta_{S1}$ and $2\theta_{S2}$

The length of OA and OB ➔ $= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$

$$\sin 2\theta_{S1} = \frac{-\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}} \qquad \text{Equation 1.15}$$

$$\cos 2\theta_{S1} = \frac{\tau_{xy}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}} \qquad \text{Equation 1.16}$$

Substituting Equation 1.15 and 1.16 into Equation 1.9

$$\tau'_{xy} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \text{Equation 1.9}$$

$$\tau_{\max} = -\frac{(\sigma_x - \sigma_y)}{2}\left[\frac{-\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}\right] + \tau_{xy}\left[\frac{\tau_{xy}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}\right]$$

$$\tau_{\max} = \frac{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$

$$\tau_{\max} = \frac{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} \times \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}}{\sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau^2{}_{xy}}}$$

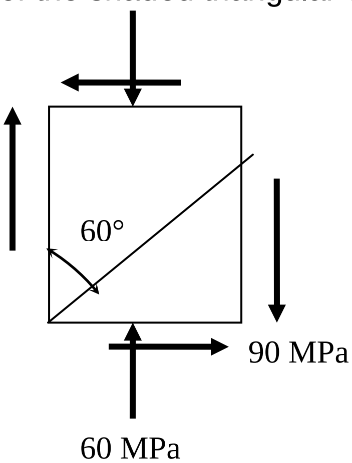
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} \qquad \text{Equation 1.17}$$

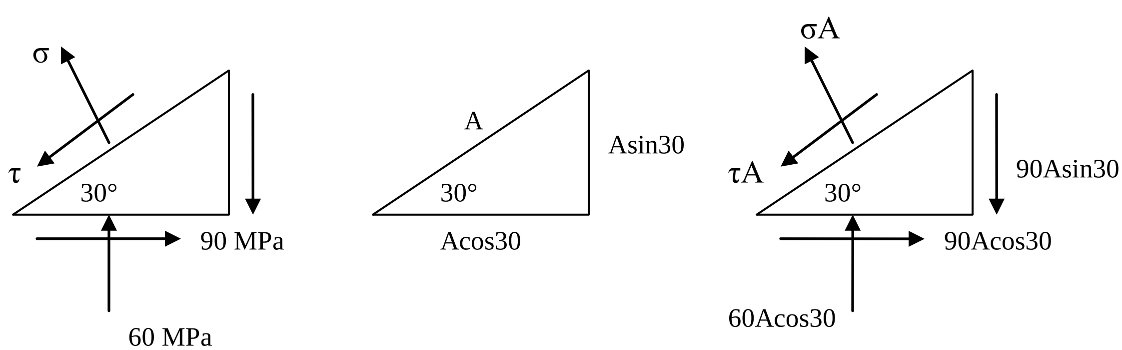
Subtituting the equation 1.15 and 1.16 into equation 1.8, we see that there is also a normal stress on the planes of maximum in-plane shear stress (σ_{ave})

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad \text{Equation 1.18}$$

Example 1.15
For the given state of stress, determine the normal stress and shearing stress exerted on the oblique face of the shaded triangular element shown**.**





+ ∑F = 0

$$\sigma A - 90A\sin 30°\cos 30° - 90A\cos 30°\sin 30° + 60A\cos 30°\cos 30° = 0$$

$$\sigma = 180\sin 30\cos 30 - 60\cos^2 30 = 32.9MPa$$

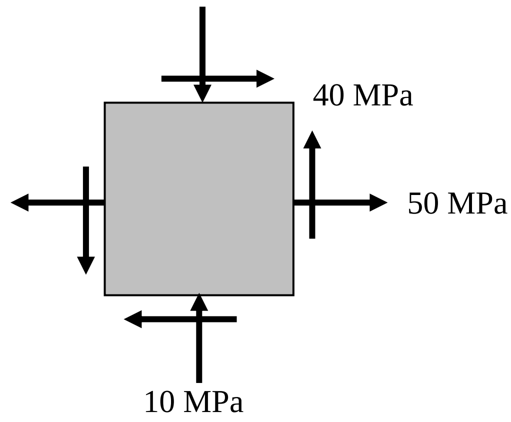
+ ∑F = 0

$$\tau A + 90A\sin 30°\sin 30° - 90A\cos 30°\cos 30° - 60A\cos 30°\sin 30° = 0$$

$$\tau = 90(\cos^2 30 - \sin^2 30) + 60\cos 30\sin 30 = 71.0MPa$$

Example 1.16

For the state of plane stress shown, determine (a) the principal stress (b) principal planes (c) maximum shear stress



(a) Principal stress

σ_x = 50 MPa σ_y = -10 MPa τ_{xy} = 40 MPa

$$\sigma_{\max,}, \sigma_{\min} = \left[\frac{\sigma_x + \sigma_y}{2}\right] \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$$

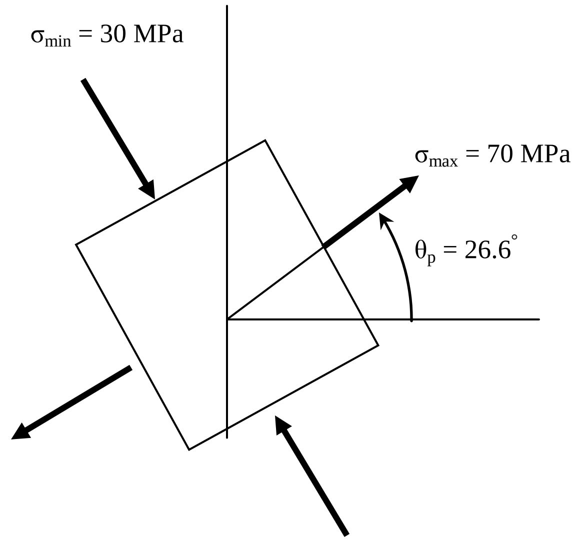
$$\sigma_{\max,}, \sigma_{\min} = \left[\frac{50 - 10}{2}\right] \pm \sqrt{\left(\frac{50 + 10}{2}\right)^2 + (40)^2}$$

$$\sigma_{\max,}, \sigma_{\min} = 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\sigma_{\max} = 20 + 50 = 70MPa$$

$$\sigma_{\min} = 20 - 50 = -30MPa$$

(b) Principal planes



$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta = \frac{2(40)}{50-(-10)} = \frac{80}{60}$$

$2\theta_p = 53.1°$ and $180° + 53.1° = 233.1°$

$\theta_p = 26.6°$ and 116.6

(c) Maximum shearing stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}} = \sqrt{(30)^2 + (40)^2} = 50MPa$$

$\theta_s = \theta_p - 45°$

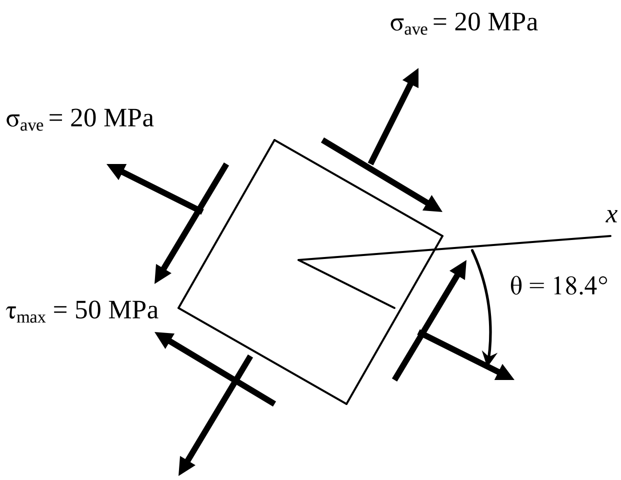
$= -18.4°$

or

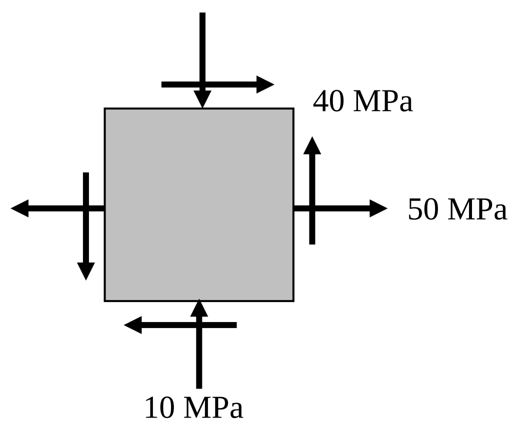
$$\tan 2\theta_S = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = \frac{-(50-(-10))}{2(40)}$$

$\theta_s = -18.4°$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50-10}{2} = 20MPa$$



Example 1.17
For the give state of stress, determine the normal and shearing stresses after the element shown has been rotated through 25° clockwise.



Solution

σ_x = 50 MPa σ_y = -10 MPa τ_{xy} = 40 MPa θ = -25°

$$\frac{\sigma_x + \sigma_y}{2} = \frac{50 + (-10)}{2} = 20MPa$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{50 - (-10)}{2} = 30MPa$$

2θ = -50°

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} = 20 + 30\cos(-50) + 40\sin(-50)$$

$$\sigma_{x'} = 8.64MPa$$

$$\sigma_{y'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] - \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = 20 - 30\cos(-50) - 40\sin(-50)$$

$$\sigma_{y'} = 50MPa$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = -(30)\sin(-50) + 40\cos 2\theta$$

$$\tau_{x'y'} = -(30)\sin(-50) + 40\cos(-50)$$

$$\tau_{x'y'} = 48.69MPa$$

1.7.2.2 Mohr's Circle Method

Mohr's circle can be used to determine the principal stresses, the maximum in-plane shear stress and average normal stress or the stress on any arbitrary plane.

Equation 1.8 and 1.9 can be defined in the form of circle as follow:

$$\sigma_{x'} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left[\frac{\sigma_x - \sigma_y}{2}\right]\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \text{Equation 1.8}$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \text{Equation 1.9}$$

By adding and squaring each equation, the θ value can be eliminated, therefore;

$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau^2{}_{x'y'} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}$$

σ_x, σ_y and τ_{xy} are constants

$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau^2{}_{x'y'} = R^2$ where $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2}$**,** therefore;

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} \qquad \text{Equation 1.19}$$

Equation 1.19 is an equation of a circle with:

$\sigma_{x'}$ and $\tau_{x'y'}$	=	coordinate on the Mohr circle,
R	=	a radius of the Mohr circle
σ_{ave} and $\tau = 0$	=	the centre of the circle.

A Mohr circle can be drawing in two manners, which are:

(a) Consider σ positive to the right and τ positive downward (orientation θ counter clockwise positive). Refer Figure 1.19(a)

(b) Consider σ positive to the right and τ positive upward (orientation θ clockwise positive). Refer Figure 1.19(b)

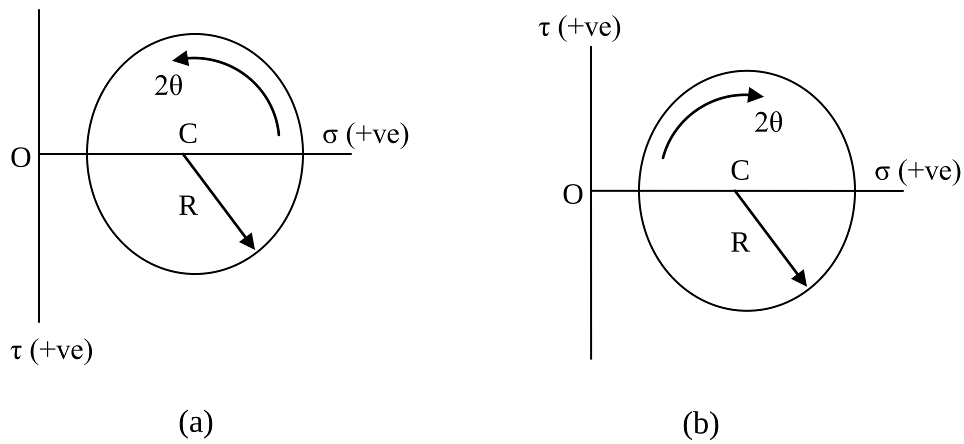


Figure 1.19: The rotation of Mohr's circle

Method (a) will be used in this chapter for solving the problems. The steps to draw the Mohr circle are:

(a) Determine the centre of the circle (point C) with coordinate $\sigma = \sigma_{ave}$ and $\tau = 0$
(b) Determine the point X at $\theta = 0$ with coordinate $X(\sigma_x, \tau_{xy})$
(c) Determine the point Y at $\theta = 90°$ with coordinate $Y(\sigma_y, -\tau_{xy})$
(d) Draw a circle which through point X and Y with centre point at C
(e) Draw line XY as a reference point

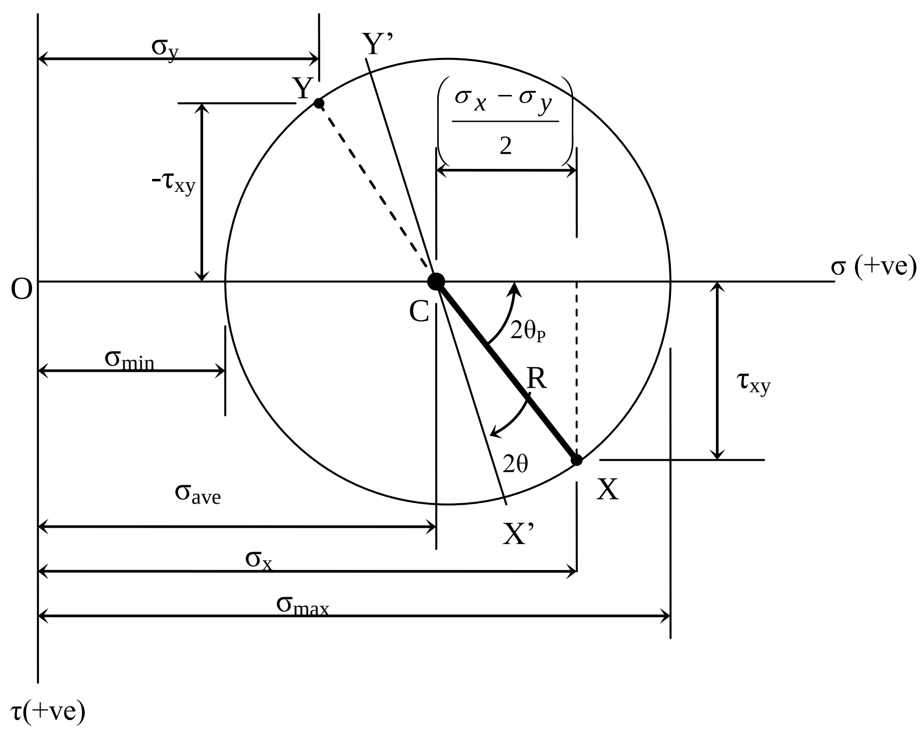
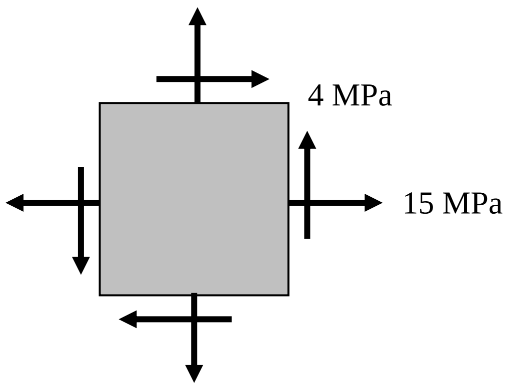


Figure 1.20: Mohr's circle

Example 1. 18

Using Mohr's circle method, determine (a) normal and shearing stresses after rotated 40° (b) principal stress (c) maximum shear stress

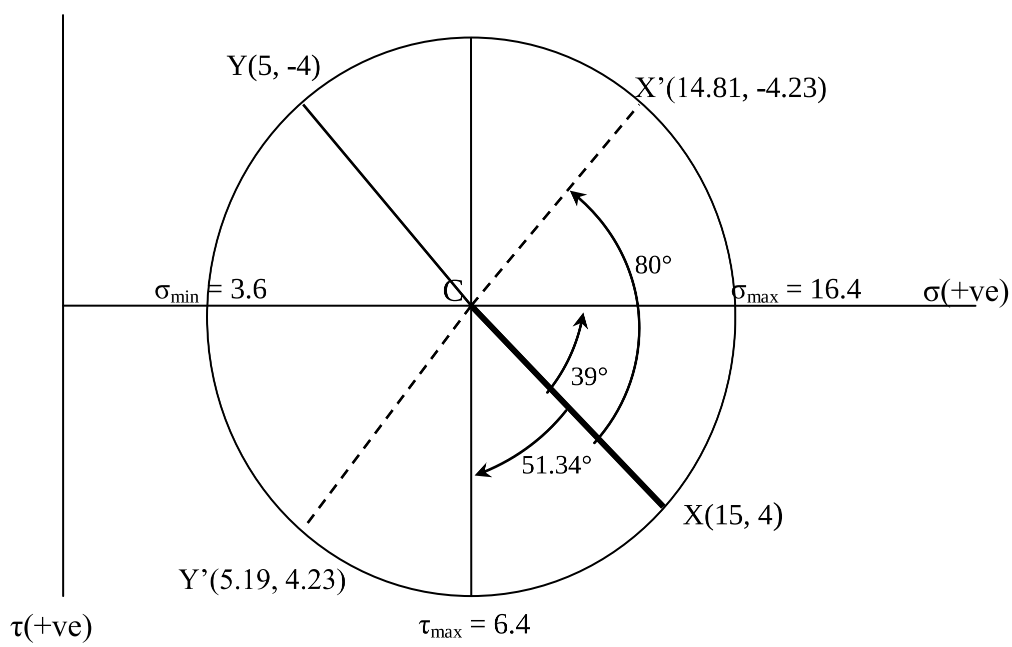


(a) dra

Point X (15, 4),   Point Y (5, -4)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{15+5}{2} = 10MPa$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} = \sqrt{5^2 + 4^2} = 6.4MPa$$



From the Mohr's circle

(a) normal and shearing stresses after rotated 40°

$\sigma_{x'}$ = 14.81 MPa, $\tau_{x'y'}$ = -4.23 MPa

$\sigma_{y'}$ = 5.19 MPa, $\tau_{y'x'}$ = 4.23 MPa

(b) Principal stress

σ_{max} = 16.40 MPa, σ_{min} = 3.6 MPa,

θ = 19.5°

(c) maximum shear stress

τ_{max} = 6.40 MPa, σ_{ave} = 10 MPa,

θ = -25.7°

Tutorial 1

A Mohr's Circle has the radius R = 52 MPa and average stress σ_{ave} = 80 MPa, respectively. Determine:

(a) Principal stress
(b) Normal and shear stress at $\theta = 0^o$ if the principal plane is 34^o.
(c) The stress component exerted on the element obtained by rotating the given element at $\theta = -30^o$.
(d) Maximum shear stress